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A SPLINE SOLUTION FOR THE (0,m) LACUNARY INTERPOLATION PROBLEM

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1. Introduction

Several authors have studied the use of splines to solve the lacunary interpolation problem. The study of this problem was initiated by Turán and Balázs in 1957 [1]. In a recent paper [6] by A. Meir and A. Sharma a spline polynomial has been constructed for (0,2) interpolation of certain function. Swartz and Varga in [8] have extended the results of [6] to a wider class of functions and have indicated that the extended results are the best possible. Th. Fawzy and F. Holail [2,3,4] have constructed spline functions for the cases (0,2), (0,3) and (0,4) and they got results of the same order as that of best approximation for the function and their all possible derivatives. Th. Fawzy and L. Schumaker solved the general lacunary interpolation problem [5], but the method introduced in this paper presents a quite different scheme and a direct one.

The paper is organized as follows. In Section 2 we construct spline polynomials for (0,5), (0,6) and (0,7) lacunary interpolation. Section 3 is devoted to the general construction of $(0,m)$ lacunary interpolation for both m-even and m-odd, which generalizes the results obtained in Section 3. In Section 4 we present two examples when $m = 11$ and $m = 12$.

2. Construction of spline polynomials for (0,5), (0,6) and (0,7) lacunary interpolation

In this section we shall study the following problem.

Problem 2.1. Given $\Delta = \{x_i = i\ h\}_{i=0}^{n}$ and two sets of real numbers $\{f_i\}$, $\{f_i^{(L)}\}$, $i = 0(1)n$ and $L = 5, 6, 7$. Find a spline interpolant $S_{\Delta,L}$ such that:

$$(2.1) \qquad S_{\Delta,L}(x_i) = f_i \quad \text{and} \quad S^{(L)}_{\Delta,L}(x_i) = f_i^{(L)}, \qquad i = 0(1)n \quad \text{and} \quad L = 5, 6, 7.$$

For each of the cases (0,5), (0,6) and (0,7) we solve Problem 2.1 by constructing a spline interpolant in the form:

$$S_{k,L}(x) = \sum_{j=0}^{L} \frac{1}{j!} S_k^{(j)} (x - x_k)^j, \tag{2.2}$$

where $x \in [x_k, x_{k+1}]$, $k = 0(1)n-1$ and $L = 5, 6, 7$.

Before giving the theorems concerning the spline interpolants $S_{\Delta,L}(x)$, we consider the following set of conditions to be fulfilled:

$$\begin{cases} S_{\Delta,L}(x) \in P_L \quad \text{on each interval } [x_k, x_{k+1}], \\ S_{\Delta,L}^{(r)}(x_k) = f_k^{(r)} \quad \text{and} \\ S_{\Delta,L} \in C^{(0,L)}[x_0, x_n], \text{ i.e. } S_{k,L}^{(r)}(x_{k+1}) = S_{k+1,L}^{(r)}(x_{k+1}) \end{cases} \tag{2.3}$$

where $k = 0(1)n-1$, $r = 0, L$ and $L = 5, 6, 7$ and P_L denotes the set of polynomials of degree at most L.

Theorem 2.1. *Suppose* $\{f_i\}$ *and* $\{f_i^{(5)}\}$ *are given real numbers associated with the knots* $\{x_i\}$, $i = 0(1)n$. *Then there exists a unique spline* $S_{\Delta,5}(x)$ *defined in (2.2) when* $L = 5$ *and satisfying the set of conditions (2.3) when* $L = 5$, *where the coefficients* $S_k^{(j)}$ *are defined as follow:*

(i) *For all* $k = 0(1)n$ *we have*

$$S_k^{(0)} = f_k, \quad S_k^{(5)} = f_k^{(5)}. \tag{2.4}$$

(ii) *For* $k = 0, 1$ *we choose*

$$\begin{cases} S_k^{(4)} = S_{k+1}^{(4)} - h\, f_k^{(5)}, \quad S_k^{(3)} = S_{k+1}^{(3)} - hS_k^{(4)} - \frac{1}{2!}h^2 f_k^{(5)}, \\ S_k^{(2)} = S_{k+1}^{(2)} - hS_k^{(3)} - \frac{1}{2!}h^2 S_k^{(4)} - \frac{1}{3!}h^3 f_k^{(5)} \quad \textit{and} \\ S_k^{(1)} = (1/h)\Delta f_k - \sum_{r=2}^{5} (1/r!) h^{r-1} S_k^{(r)}. \end{cases} \tag{2.5}$$

(iii) *For $k = 2, 3, \ldots, n-2$ we get*

$$(2.6)\qquad \begin{cases} S_k^{(4)} = (1/h^4)\Delta^4 f_{k-2}, \\ S_k^{(3)} = (1/h^3)\Delta^3 f_{k-1} - \frac{12}{4!} h S_k^{(4)} - \frac{30}{5!} h^2 f_k^{(5)}, \\ S_k^{(2)} = (1/h^2)\Delta^2 f_{k-1} - \frac{2}{4!} h^2 S_k^{(4)}, \\ S_k^{(1)} = (1/h)\Delta f_k - \sum_{r=2}^{5} (1/r!) h^{r-1} S_k^{(r)}. \end{cases}$$

(iv) *For $k = n-1$ we take*

$$(2.7)\qquad S_k(x) = S_{k-1}(x_k) + \sum_{r=1}^{5} (1/r!) S_{k-1}^{(r)}(x_k)(x - x_k)^r.$$

Clearly, the function $S_{\Delta,5}$ defined in (2.2) - (2.7) exists and is unique.

Theorem 2.2. *Suppose $\{f(x_i),\ f^{(6)}(x_i)\}$ are given real numbers, $i =$ $= 0(1)n-1$. Then, there exists a unique spline function $S_{\Delta,6(x)}$ which is given in (2.2) and satisfies (2.3) when $L = 6$, where the coefficients $S_k^{(j)}$ are defined as follow:*

$$(2.8)\qquad S_k = f_k \quad \text{and} \quad S_k^{(6)} = f_k^{(6)}, \quad k = 0(1)n-1$$

and for the other coefficients we have the following cases:

Case 1. *For $k = 0, 1$ we choose*

$$(2.9)\qquad \begin{cases} S_k^{(5)} = S_{k+1}^{(5)} - h f_k^{(6)}, \quad S_k^{(4)} = S_{k+1}^{(4)} - h S_k^{(5)} - \frac{1}{2!} h^2 f_k^{(6)}, \\ S_k^{(3)} = S_{k+1}^{(3)} - h S_k^{(4)} - \frac{1}{2!} h^2 S_k^{(5)} - \frac{1}{3!} h^3 f_k^{(6)}, \\ S_k^{(2)} = S_{k+1}^{(2)} - \sum_{r=3}^{5} \frac{1}{(r-2)!} h^{r-2} S_k^{(r)} - \frac{1}{4!} h^4 f_k^{(6)} \quad \text{and} \\ S_k^{(1)} = (1/h)\Delta f_k - \sum_{r=2}^{6} (1/r!) h^{r-1} S_k^{(r)}. \end{cases}$$

Case 2. *For $k = 2(1)n-3$ we obtain*

$$(2.10)\qquad \begin{cases} S_k^{(5)} = (1/h^5)\Delta^5 f_{k-2} - \dfrac{360}{6!} h f_k^{(6)}, \\ S_k^{(4)} = (1/h^4)\Delta^4 f_{k-2} - \dfrac{120}{6!} h f_k^{(6)}, \\ S_k^{(3)} = (1/h^3)\Delta^3 f_{k-1} - \dfrac{12}{4!} h f_k^{(4)} - \dfrac{30}{5!} h^2 f_k^{(5)} - \dfrac{60}{6!} h^3 f_k^{(6)}, \\ S_k^{(2)} = (1/h^2)\Delta^2 f_{k-1} - \dfrac{2}{4!} h^2 f_k^{(4)} - \dfrac{2}{6!} h^4 f_k^{(6)} \quad \text{and} \\ S_k^{(1)} = (1/h)\Delta f_k - (1/h) \sum_{r=2}^{6} (1/r!) h^r f_k^{(r)}. \end{cases}$$

Case 3. *For $k = n-2,\ n-1$ we take*

$$(2.11)\qquad S_k(x) = S_{k-1}(x_k) + \sum_{r=1}^{6} (1/r!) S_{k-1}^{(r)}(x_k)(x - x_k)^r.$$

Clearly, the function $S_{\Delta,6}$ defined in (2.8) - (2.11) exists and is unique.

Theorem 2.3. *Given two sets of arbitrary values of the function $f(x)$ and $f^{(7)}(x)$ at the knots $x = x_k$, $k = 0(1)n$, then there exists a unique spline polynomial $S_{\Delta,7}(x)$ defined by (2.2) when $L = 7$ and satisfies the set of conditions (2.3) when $L = 7$, where $S_k^{(7)}$ are defined as follows:*

(i) *For all $k = 0(1)n$ we have*

$$(2.12)\qquad S_k^{(7)} = f_k^{(7)}, \quad S_k^{(0)} = f_k.$$

(ii) *For $k = 0, 1, 2$ we find that*

$$(2.13)\qquad \begin{cases} S_k^{(1)} = (1/h)[f_{k+1} - f_k] - \sum_{r=2}^{7} (1/r!) h^{r-1} S_k^{(r)}, \\ S_k^{(j)} = S_{k+1}^{(j)} - \sum_{r=j+1}^{7} [1/(r-j)!] h^{r-j} S_k^{(r)}, \quad j = 2(1)6. \end{cases}$$

(iii) *For $k = 3(1)n-3$ we get*

$$
(2.14)\qquad
\begin{cases}
S_k^{(1)} = (1/h)\Delta f_k - 1!\sum_{r=2}^{7}(1/r!)h^{r-1}S_k^{(r)},\\
S_k^{(2)} = (1/h^2)\Delta^2 f_{k-1} - 2!\left\{\frac{1}{4!}h^2S_k^{(4)} + \frac{1}{6!}h^4S_k^{(6)}\right\},\\
S_k^{(3)} = (1/h^3)\Delta^3 f_{k-1} -\\
\qquad -3!\left\{\frac{2}{4!}hS_k^{(4)} + \frac{5}{5!}h^2S_k^{(5)} + \frac{10}{6!}h^3S_k^{(6)} + \frac{21}{7!}h^4S_k^{(7)}\right\},\\
S_k^{(4)} = (1/h^4)\Delta^4 f_{k-2} - 4!\left\{\frac{5}{6!}h^2S_k^{(6)}\right\},\\
S_k^{(5)} = (1/h^5)\Delta^5 f_{k-2} - 5!\left\{\frac{3}{6!}hS_k^{(6)} + \frac{14}{7!}h^2S_k^{(7)}\right\} \quad \textit{and}\\
S_k^{(6)} = (1/h^6)\Delta^6 f_{k-3}.
\end{cases}
$$

(iv) *For $k = n-2,\ n-1$ we choose*

$$
(2.15)\qquad
\begin{cases}
S_k^{(r)} = S_{k-1}^{(j)}(x_k), \quad r = j = 0(1)7,\\
S_{k-1}^{(j)}(x_k) = \sum_{r=j}^{7}[1/(r-j)!]h^{r-j}S_{k-1}^{(r)}, \quad k = n-2,\\
S_{k-1}^{(j)}(x_k) = \sum_{r=j}^{7}[1/(r-j)!]h^{r-j}S_{k-2}^{(r)}(x_{k-1}), \quad k = n-1.
\end{cases}
$$

Clearly, $S_{\Delta,7}(x)$ defined in (2.12) - (2.15) exists and is unique.

3. General construction of (0,m)-lacunary interpolant

In this section we shall introduce the general construction of $(0,m)$-lacunary interpolation polynomial. Now we study the $(0,m)$ interpolation problem.

Problem 3.1. Let $\Delta : 0 = x_0 < x_1 < x_2 < \ldots < x_n = 1$. Let m be a positive integer, $m \geq 2$. Suppose that $\{f_i,\ f_i^{(m)}\}$ are given real numbers, $i = 0(1)n$. Find a spline function $S_{\Delta,m}$ such that

$$
(3.1)\qquad D^j S_{\Delta,m}(x_i) = f_i^{(j)}, \quad i = 0(1)n, \quad j = 0 \ \text{and} \ m.
$$

For solving this problem we shall construct a spline polynomial in the following form:

$$S_{\Delta,m}(x) = \sum_{j=0}^{m}(1/j!)S_k^{(j)}(x - x_k)^j, \quad m \geq 2. \tag{3.2}$$

Here we consider

$$\begin{cases} S_{\Delta,m}(x) \in P_m \quad \text{on each} \quad x \in [x_k,\ x_{k+1}] \\ S_{\Delta,m}^{(r)}(x_k) = f_k^{(r)} \\ S_{\Delta,m} \in C^{(0,m)}[x_0,\ x_n], \end{cases} \tag{3.3}$$

where $k = 0(1)n - 1,\ r = 0$ and m and $m \geq 2$.

Now we introduce the following theorem concerning the construction of the general formula $(0, m)$.

Theorem 3.1. *Given two sets of arbitrary values of the function $f^{(j)}(x),\ j = 0$ and $m,\ m \geq 2$, at the knots $x = x_k,\ k = 0(1)n$. Then there exists a unique spline function $S_{\Delta,m}(x)$ defined in (3.2) and satisfying the set of conditions (3.3) where the coefficients $S_k^{(i)}$ are defined as follow:*

$$S_k^{(0)} = f_k \quad \text{and} \quad S_k^{(m)} = f_k^{(m)}, \quad k = 0(1)n \quad \text{and} \tag{3.4}$$

for the other coefficients we have the following cases:

Case 1.

(i) *When i is odd and* $k = \frac{m-1}{2},\ \frac{m-1}{2} + 1,\ \frac{m-1}{2} + 2, \ldots, \frac{m-1}{2} + (i - m + 1)$.

The general formula for the coefficients is

$$S_k^{(i)} = (1/h^i)\Delta^i f_{k-(\frac{i-1}{2})} - i!\left\{ \frac{C_{i,1}}{(i+1)!}hS_k^{(i+1)} + \frac{C_{i,2}}{(i+2)!}h^2S_k^{(i+2)} + \ldots + \frac{C_{i,r}}{(i+r)!}h^r S_k^{(i+r)} \right\}, \tag{3.5}$$

where

$$C_{i,1} = \frac{i+1}{2}, \qquad i = 1,3,5,\dots$$
$$C_{1,r} = 1, \qquad r = 1,2,3,\dots$$
$$C_{i,r} = \left(\frac{i+1}{2}\right) C_{i,r-1} + C_{i-2,r}, \qquad r \text{ is even}$$
$$= \left(\frac{i+1}{2}\right) C_{i,r-1} \qquad r \text{ is odd}$$

$i = 3,5,\dots,m-1$ (*m is even*), $i = 3,5,\dots,m-2$ (*m is odd*) *and* $r = 2,3,\dots,m-i$.

(ii) *When* i *is even and* $k = \frac{m}{2}-1, \frac{m}{2}, \frac{m}{2}+1, \dots, \frac{m}{2}+(i-m)$.

Then the general formula for the coefficients will be:

$$S_k^{(i)} = (1/h^i)\Delta^i f_{k-\frac{i}{2}} - i!\left\{ \frac{C_{i,1}}{(i+2)!}h^2 S_k^{(i+2)} + \right.$$
$$\left. + \frac{C_{i,2}}{(i+4)!}h^4 S_k^{(i+4)} + \dots + \frac{C_{i,r}}{(i+2r)!}h^{2r} S_k^{(i+2r)} \right\}, \tag{3.6}$$

where

$$C_{i,r} = C_{i-1,2r},$$

$i = 2,4,\dots,m-2$ (*m is even*), $i = 2,4,\dots,m-1$ (*m is odd*) *and*
$r = 1,2,\dots,\frac{m-i}{2}$ (*m is even*), $r = 1,2,\dots,\frac{m-i-1}{2}$ (*m is odd*).

Case 2.

(i) *For* $k = 0,1,\dots,\frac{m}{2}-2$ *when m is even and* $k = 0,1,\dots,\frac{m-1}{2}-1$ *when m is odd.*

Then we have the following general formula for the coefficients:

$$\begin{cases} S_k^{(1)} = (1/h)[f_{k+1} - f_k] - \sum_{r=2}^{m}(1/r!)h^{r-1}S_k^{(r)}, \\ S_k^{(i)} = S_{k+1}^{(i)} - \sum_{r=i+1}^{m}[1/(r-n)!]h^{r-i}S_k^{(r)}, \quad i = 2,3,\dots,m-1. \end{cases} \tag{3.7}$$

(ii) *For* $k = i-1, i-2, i-3, \ldots, \left[\frac{m}{2} + (i-m-1)\right]$ *when m is even,* $k = n-1, n-2, n-3, \ldots, \left[\frac{m-1}{2} + (i-m)\right]$ *when m is odd.*

$$(3.8)\quad \begin{cases} S_k(x) = \sum_{r=0}^{m} (1/r!) S_k^{(r)} (x - x_k)^r, \\ S_k^{(r)} = S_{k-1}^{(j)}(x_k), \quad r = j = 0(1)m, \\ S_{k-1}^{(j)}(x_k) = \sum_{r=j}^{m} [1/(r-j)!] h^{r-j} S_{k-1}^{(r)}, \\ k = \frac{m}{i} + (i-m-1)\ (m-even)\ and\ k = \frac{m-1}{2} + (i-m)\ (m-odd); \\ S_{k-1}^{(j)}(x_k) = \sum_{r=j}^{m} [1/(r-j)!] h^{r-j} S_{k-2}^{(r)}(x_{k-1}), \\ k = i-1, i-2, i-3, \ldots, \frac{m}{2} + (i-m), \quad (m-even) \quad and \\ k = i-1, i-2, i-3, \ldots, \frac{m-1}{2} + (i-m-1) \quad (m-odd). \end{cases}$$

4. Examples

By using the general formula obtained in Section 3 we give the following examples when $m = 11$ and $m = 12$.

(a) When m=11. We have the following construction of the spline interpolant $S_{\Delta,11}$ defined by (3.2) when $m = 11$, where the coefficients $S_k^{(j)}$ are defined by using theorem 3.1 as follows:

For $k = 0(1)4$ *we have*

$$S_k^{(1)} = (1/h)\Delta f_k - \sum_{r=2}^{11} (1/r!) h^{r-1} S_k^{(r)},$$

$$S_k^{(i)} = S_{k+1}^{(i)} - \sum_{r=i+1}^{11} [1/(r-i)!] h^{r-i} S_k^{(r)}, \quad i = 2(1)10.$$

For $k = 5(1)n-5$ *we have*

$$S_k^{(1)} = (1/h)\Delta f_k - 1! \sum_{r=2}^{11} (1/r!) h^{r-1} S_k^{(r)},$$

$$S_k^{(2)} = (1/h^2)\Delta^2 f_{k-1} - 2! \sum_{r=4,6,\ldots}^{10} (1/r!) h^{r-2} S_k^{(r)},$$

$$S_k^{(3)} = (1/h^3)\Delta^3 f_{k-1} - 3!\Big[(2/4!)hS_k^{(4)} + (5/5!)S_k^{(5)} + \\ + (10/6!)h^3 S_k^{(6)} + (21/7!)h^4 S_k^{(7)} + (42/8!)h^5 S_k^{(8)} + \\ + (85/9!)h^6 S_k^{(9)} + (170/10!)h^7 S_k^{(10)} + (341/11!)h^8 S_k^{(11)}\Big],$$

$$S_k^{(4)} = (1/h^4)\Delta^4 f_{k-2} - 4!\Big[(5/6!)h^2 S_k^{(6)} + (21/8!)h^4 S_k^{(8)} + \\ + (85/10!)h^6 S_k^{(10)}\Big],$$

$$S_k^{(5)} = (1/h^5)\Delta^5 f_{k-2} - 5!\Big[(3/6!)hS_k^{(6)} + (14/7!)h^2 S_k^{(7)} + \\ + (42/8!)h^3 S_k^{(8)} + (147/9!)h^4 S_k^{(9)} + (441/10!)h^5 S_k^{(10)} + \\ + (1408/11!)h^6 S_k^{(11)}\Big],$$

$$S_k^{(6)} = (1/h^6)\Delta^6 f_{k-3} - 6!\Big[(14/8!)h^2 S_k^{(8)} + (147/10!)h^4 S_k^{(10)}\Big],$$

$$S_k^{(7)} = (1/h^7)\Delta^7 f_{k-3} - 7!\Big[(4/8!)hS_k^{(8)} + (30/9!)h^2 S_k^{(9)} + \\ + (120/10!)h^3 S_k^{(10)} + (627/11!)h^4 S_k^{(11)}\Big],$$

$$S_k^{(8)} = (1/h^8)\Delta^8 f_{k-4} - 8!\Big[(30/10!)h^2 S_k^{(10)}\Big],$$

$$S_k^{(9)} = (1/h^9)\Delta^9 f_{k-4} - 9!\Big[(5/10!)hS_k^{(10)} + (55/11!)h^2 S_k^{(11)}\Big] \quad \text{and}$$

$$S_k^{(10)} = (1/h^{10})\Delta^{10} f_{k-5},$$

and

For $k = n-4(1)n-1$

$$S_k^{(r)} = S_{k-1}^{(j)}(x_k), \quad r = j = 0(1)11,$$

$$S_{k-1}^{(j)}(x_k) = \sum_{r=j}^{11}[1/(r-j)!]h^{r-j} S_{k-1}^{(r)}, \quad k = n-4,$$

$$S_{k-1}^{(j)}(x_k) = \sum_{r=j}^{11}[1/(r-j)!]h^{r-j} S_{k-2}^{(r)}(x_{k-1}), \quad k = n-3(1)n-1.$$

(b) When m=12. We have the following construction of the spline interpolant defined by (3.2) when $m = 12$ and by using Theorem 3.1 we could easily define the coefficients $S_k^{(j)}$ as follows:

For $k = 0(1)4$

$$S_k^{(1)} = (1/h)\Delta f_k - \sum_{r=2}^{12}(1/r!)h^{r-1}S_k^{(r)},$$

$$S_k^{(i)} = S_{k+1}^{(i)} - \sum_{r=i+1}^{12}[1/(r-n)!]h^{r-n}S_k^{(r)}, \quad n = 2(1)11.$$

For $k = 5(1)n-6$

$$S_k^{(1)} = (1/h)\Delta f_k - 1!\sum_{r=2}^{12}(1/r!)h^{r-1}S_k^{(r)},$$

$$S_k^{(2)} = (1/h^2)\Delta^2 f_{k-1} - 2!\sum_{r=4,6,\dots}^{12}(1/r!)h^{r-2}S_k^{(r)},$$

$$S_k^{(3)} = (1/h^3)\Delta^3 f_{k-1} - 3!\Big[(2/4!)hS_k^{(4)} + (5/5!)h^2S_k^{(5)} + (10/6!)h^3S_k^{(6)} + (21/7!)h^4S_k^{(7)} + (42/8!)h^5S_k^{(8)} + (85/9!)h^6S_k^{(9)} + (170/10!)h^7S_k^{(10)} + (341/11!)h^8S_k^{(11)} + (682/12!)h^9S_k^{(12)}\Big],$$

$$S_k^{(4)} = (1/h^4)\Delta^4 f_{k-2} - 4!\Big[(5/6!)h^2S_k^{(6)} + (21/8!)h^4S_k^{(8)} + (85/10!)h^6S_k^{(10)} + (341/12!)h^8S_k^{(12)}\Big],$$

$$S_k^{(5)} = (1/h^5)\Delta^5 f_{k-2} - 5!\Big[(3/6!)hS_k^{(6)} + (14/7!)h^2S_k^{(7)} + (42/8!)h^3S_k^{(8)} + (147/9!)h^4S_k^{(9)} + (441/10!)h^5S_k^{(10)} + (1408/11!)h^6S_k^{(11)} + (4224/12!)h^7S_k^{(12)}\Big],$$

$$S_k^{(6)} = (1/h^6)\Delta^6 f_{k-3} - 6!\Big[(14/8!)h^2S_k^{(8)} + (147/10!)h^4S_k^{(10)} + (1408/12!)h^6S_k^{(12)}\Big],$$

$$S_k^{(7)} = (1/h^7)\Delta^7 f_{k-3} - 7!\Big[(4/8!)hS_k^{(8)} + (30/9!)h^2S_k^{(9)} + (120/10!)h^3S_k^{(10)} + (627/11!)h^4S_k^{(11)} + (2508/12!)h^5S_k^{(12)}\Big],$$

$$S_k^{(8)} = (1/h^8)\Delta^8 f_{k-4} - 8!\Big[(30/10!)h^2 S_k^{(10)} + (627/12!)h^4 S_k^{(12)}\Big],$$

$$S_k^{(9)} = (1/h^9)\Delta^9 f_{k-4} - 9!\Big[(5/10!)hS_k^{(10)} + (55/11!)h^2 S_k^{(11)} + + (275/12!)h^3 S_k^{(12)}\Big],$$

$$S_k^{(10)} = (1/h^{10})\Delta^{10} f_{k-5} - 10!\Big[(55/12!)h^2 S_k^{(12)}\Big] \quad \text{and}$$

$$S_k^{(11)} = (1/h^{11})\Delta^{11} f_{k-5} - 11!\Big[(6/12!)hS_k^{(12)}\Big]$$

and

For $k = n - 5(1)n - 1$

$$S_k^{(r)} = S_{k-1}^{(j)}(x_k), \quad r = j = 0(1)12,$$

$$S_{k-1}^{(j)}(x_k) = \sum_{r=0}^{12}[1/(r-j)!]h^{r-j}S_{k-1}^{(r)}, \quad k = n - 5,$$

$$S_{k-1}^{(j)}(x_k) = \sum_{r=j}^{12}[1/(r-j)!]h^{r-j}S_{k-2}^{(r)}(x_{k-1}) \quad k = n - 4(1)n - 1.$$

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