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AN ASYMPTOTIC APPROACH TO THE MULTIPLE MACHINE INTERFERENCE PROBLEM WITH MARKOVIAN ENVIRONMENTS*

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Abstract. This paper is concerned with a queueing model to analyse the asymptotic behaviour of the machine interference problem with N machines and n operatives. The machines and the repair facility are assumed to operate in independent random environments governed by ergodic Markov chains. The running and repair times of machines are supposed to be exponentially distributed random variables with parameter depending on the number of stopped machines and the state of the corresponding varying environment. Assuming that the repair rate is much greater than the failure rate ("fast" service), it is shown that the time until the number of stopped machines reaches a certain level converges weakly, under appropriate norming, to an exponentially distributed random variable. Furthermore, some numerical examples illustrate the problem in question in the field of textile winding.

1. Introduction

The machine interfence model has been considered by a number of authors. In its simplest form, where there are exponential running and repair times, a fixed number of machines in the system and a fixed number of repairmen, the problem is frequently used as a textbook example of a birth-death model or a finite-source exponential queueing system. Many articles have generalised this basic model by assuming, for example, general repair times, general operating times, etc. For an updated review see Agnihothri [1], Carmichael [6], Stecke and Aronson [12]. In recent years this model has been used, for example, for the mathematical description of computer terminal systems, cf. Takagi [15], or for modelling production systems in textile winding, see Bunday

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[4]. More recently several authors have tackled the problem for non-identical set of machines. Among those contributing are Bunday and Khorram [5], Sztrik [13,14], Tosirisuk and Chandra [16] in which an extensive bibliography can be found on this topic. In these papers the main aim has been to predict the steady-state operational measures, such as machine availability, operative utilization, mean waiting time, average queue length. The diffusion approximation, cf. Sivazlian and Wang [11], is based on the assumption that the queue of failed machines is almost always nonempty, that is, we have a heavy traffic situation. In this study another asymptotic approach, namely a light traffic approximation is presented to analyse the distribution of the time until the number of stopped machines reaches a certain level. This method is quite common in reliability theory; see among others Anisimov and Sztrik [3], Gertsbakh [8,9], Keilson [11]. Refinements in the model are often needed when the system environment is subject to randomly occuring fluctuations which appear as changes in the parameters of the model. These fluctuations may be due to the weather, earthquakes, or other changes in the physical environment, to personnel changes, to alteration of system usage intensity, etc., see Gaver et al. [7].

This paper is concerned with a queueing model to analyse the asymptotic behaviour of the machine interference problem with N machines and n operatives. The machines and the repair facility are assumed to operate in independent random environments governed by ergodic Markov chains. The running and repair times of machines are supposed to be exponentially distributed random variables with parameter depending on the number of stopped machines and the state of the corresponding varying environment. Assuming that the repair rate is much greater than the failure rate ("fast" service), it is shown that the time until the number of stopped machines reaches a certain level converges weakly, under appropriate norming, to an exponentially distributed random variable. Furthermore, some numerical examples illustrate the problem in question in the field of textile winding.

2. Preliminary results

In this section a brief survey is given of the most related theoretical results, mainly due to Anisimov, to be applied later on.

Let $(X_\epsilon(k), k \geq 0)$ be a Markov chain with state space

$$\bigcup_{q=0}^{m+1} X_q, \qquad X_i \cap X_j = 0, \ i \neq j,$$

defined by the transition matrix $\|p_\epsilon(i^{(q)}, j^{(z)})\|$ satisfying the following conditions:

1. $p_\varepsilon(i^{(0)}, j^{(0)}) \to p_0(i^{(0)}, j^{(0)})$, as $\varepsilon \to 0$, $i^{(0)}, j^{(0)} \in X_0$, and $P_0 = \|p_0(i^{(0)}, j^{(0)})\|$ is irreducible;
2. $p_\varepsilon(i^{(q)}, j^{(q+1)}) = \varepsilon\alpha^{(q)}(i^{(q)}, j^{(q+1)}) + o(\varepsilon)$, $i^{(q)} \in X_q$, $j^{(q+1)} \in X_{q+1}$;
3. $p_\varepsilon(i^{(q)}, f^{(q)}) \to 0$, as $\varepsilon \to 0$, $i^{(q)}$, $f^{(q)} \in X_q$, $q \geq 1$;
4. $p_\varepsilon(i^{(q)}, f^{(z)}) \equiv 0$, $i^{(q)} \in X_q$, $f^{(z)} \in X_z$, $z - q \geq 2$.

In the sequel the set of states X_q is called the q-th level of the chain, $q = 1, \ldots, m+1$. Let us single out the subset of states

$$\langle \alpha_m \rangle = \bigcup_{q=0}^{m} X_q.$$

Denote by $\{\pi_\varepsilon(i^{(q)}),\ i^{(q)} \in X_q\}$, $q = 1, \ldots, m$ the stationary distribution of a chain with transition matrix

$$\left\| \frac{p_\varepsilon(i^{(q)}, j^{(z)})}{1 - \sum\limits_{k^{(m+1)} \in X_{m+1}} p_\varepsilon(i^{(q)}, k^{(m+1)})} \right\|, i^{(q)} \in X_q, j^{(z)} \in X_z,\ \ q, z \leq m.$$

Furthermore denote by $g_\varepsilon\ (\langle \alpha_m \rangle)$ the steady state probability of exit from $\langle \alpha \rangle$, that is

$$g_\varepsilon(\langle \alpha_m \rangle) = \sum_{i^{(m)} \in X_m} \ \sum_{j^{(m+1)} \in X_{m+1}} p_\varepsilon(i^{(m)}, j^{(m+1)}).$$

Denote by $\{\pi_0(i^{(0)}),\ i^{(0)} \in X_0\}$ the stationary distribution corresponding to P_0 and let

$$\bar{\pi}_0 = \{\pi_0(i^{(0)}), i^{(0)} \in X_0\}, \quad \bar{\pi}_\varepsilon^{(q)} = \{\pi_\varepsilon(i^{(q)}), i^{(q)} \in X_q\},$$

be row vectors. Finally, let

$$A^{(q)} = \|\alpha^{(q)}(i^{(q)}, j^{(q+1)})\|, \quad i^{(q)} \in X_q,\ j^{(q+1)} \in X_{q+1}, \quad q = 0, \ldots, m$$

defined by condition 2. Conditions 1–4 enable us to compute the main terms of the asymptotic expression for $\bar{\pi}_\varepsilon^{(q)}$ and $g_\varepsilon(\langle \alpha \rangle)$. Namely, we obtain

$$\begin{aligned} \bar{\pi}_\varepsilon^{(q)} &= \varepsilon^q \bar{\pi}_0 A^{(0)} A^{(1)} \ldots A^{(q-1)} + o(\varepsilon^q), \quad q = 1, \ldots, m, \\ g_\varepsilon(\langle \alpha_m \rangle) &= \varepsilon^{m+1} \bar{\pi}_0 A^{(0)} A^{(1)} \ldots A^{(m)} \underline{1} + o(\varepsilon^{m+1}), \end{aligned} \tag{1}$$

where $1 = (1, \ldots, 1)$ is a column vector, see Anisimov et al. [2] pp. 141–153.

Let $(\eta_\varepsilon(t),\ t \geq 0)$ be a Semi Markov Process (SMP) given by the embedded Markov chain $(X_\varepsilon(k),\ k \geq 0)$ satisfying conditions 1–4. Let the times $\tau_\varepsilon(j^{(s)}, k^{(z)})$– transition times from state $j^{(s)}$ to state $k^{(z)}$– fulfil the condition

$$E\exp\{i\theta\beta_\varepsilon\tau_\varepsilon(j^{(s)}, k^{(z)})\} = 1 + a_{jk}(s, z, \theta)\varepsilon^{m+1} + o(\varepsilon^{m+1}), \quad (i^2 = -1)$$

where β_ε is some normalizing factor. Denote by $\Omega_\varepsilon(m)$ the instant at which the SMP reaches the $(m+1)$-th level for the first time, exit time from $\langle\alpha_m\rangle$, provided $\eta_\varepsilon(0) \in \langle\alpha_m\rangle$. Then we have:

Theorem 1. *(cf. Anisimov et al.* [2] *pp. 153) If the above conditions are satisfied then*

$$\lim_{\varepsilon\to 0} E\exp\{i\theta\beta_\varepsilon\Omega_\varepsilon(m)\} = (1 - A(\theta))^{-1},$$

where

$$A(\theta) = \frac{\sum\limits_{j^{(0)},k^{(0)}\in X_0} \pi_0(j^{(0)}p_0(j^{(0)}, k^{(0)})a_{jk}(0,0,\theta)}{\bar{\pi}_0 A^{(0)}A^{(1)}\dots A^{(m)}\underline{1}}.$$

Corollary 1. *In particular, if* $a_{jk}(s,z,\theta) = i\theta m_{jk}(s,z)$ *then the limit is an exponentially distributed random variable with mean*

$$\frac{\sum\limits_{j^{(0)},k^{(0)}\in X_0} \pi_0(j^{(0)}p_0(j^{(0)}, k^{(0)}m_{jk}(0,0)}{\bar{\pi}_0 A^{(0)}A^{(1)}\dots A^{(m)}\underline{1}}.$$

3. The queueing model

Let us consider the machine interference problem with N machines which are looked after by n operatives. The machines are assumed to operate in a random environment governed by an ergodic Markov chain $(\xi_1(t),\ t \geq 0)$ with state space $(1, \dots, r_1)$ and with transition density matrix $(a_{ij},\ i,j = 1, \dots, r_1,\ a_{ii} = \sum_{j\neq i} a_{ij})$. Whenever the environmental process is in state i and there are s machines stopped, $s = 0, \dots, N-1$, the probability that an operating machine breaks down in the time interval $(t, t+h)$ is $\lambda(i,s)h + o(h)$. Each machine is immediately repaired if there is an idle operative, otherwise a queueing line is formed. The service discipline is First Come-First Served (FCFS). The repair facility is also supposed to operate in a random environment governed by an ergodic Markov chain $(\xi_2(t),\ t \geq 0)$ with state space $(1, \dots, r_2)$ and with transition density matrix $(b_{kq},\ k,q = 1, \dots, r_2,\ b_{kk} = \sum_{q\neq k} b_{kq})$. Whenever the environmental process is in state k and there are s machines stopped,

$s = 1, \dots, N$, the probability that the repair of a given machine is completed in time interval $(t, t+h)$ is $\mu(k, s; \varepsilon)h + o(h)$. After being repaired each machine immediately starts operating. All random variables involved here and the random environments are supposed to be independent of each other.

Let us consider the system under the assumption of "fast" repair, i.e., $\mu(k, s; \varepsilon) \to \infty$ as $\varepsilon \to 0$. For simplicity let $\mu(k, s; \varepsilon) = \mu(k, s)/\varepsilon$.
Denote by $Y_\varepsilon(t)$ the number of stopped machines at time t, and let

$$\Omega_\varepsilon(m) = \inf\{t : \ t > 0, \ Y_\varepsilon(t) = m+1 \mid Y_\varepsilon(0) \le m\},$$

that is, the instant at which the number of stopped machines reaches the $(m+1)$–th level for the first time, provided that at the beginning their number is not greater than m; $m = 1, \dots, N-1$.

Denote by $(\pi_i^{(1)}, \ i = 1, \dots, r_1)$, $(\pi_k^{(2)}, \ k = 1, \dots, r_2)$ the steady-state distribution of the governing Markov chains $(\xi_1(t), \ t \ge 0)$, $(\xi_2(t), \ t \ge 0)$, respectively. Now we have:

Theorem 2. *For the system in question under the above assumptions, independently of the initial state, the distribution of the normalized random variable* $\varepsilon^m \Omega_\varepsilon(m)$ *converges weakly to an exponentially distributed random variable with parameter*

$$\Lambda = N \sum_{i=1}^{r_1} \sum_{k=1}^{r_2} \pi_i^{(1)} \pi_k^{(2)} \lambda(i, o) \prod_{s=1}^{m} \frac{(N-s)\lambda(i,s)}{\min(n,s)\mu(k,s)}.$$

Proof. It is easy to see that the process

$$Z_\varepsilon(t) = (\xi_1(t), \xi_2(t), Y_\varepsilon(t))$$

is a three-dimensional Markov chain with state space

$$((i,k,s), \ i = 1, \dots, r_1, \ k = 1, \dots, r_2, \ s = 0, \dots, N).$$

Furthermore, let

$$\langle \alpha_m \rangle = ((i,k,s), \ i = 1, \dots, r_1, \ k = 1, \dots, r_2, \ s = 0, \dots, m).$$

Hence our aim is to determine the distribution of the first exit time of $Z_\varepsilon(t)$ from $\langle \alpha_m \rangle$, provided that $Z_\varepsilon(o) \in \langle \alpha_m \rangle$.
It can easily be verified that the transition probabilities in any time interval $(t, t+h)$ are the following:

$$(i,k,s) \xrightarrow{h} \begin{cases} (j,k,s) & a_{ij}h + o(h), & i \ne j, \\ (i,q,s) & b_{kq} + o(h), & k \ne q, \\ (i,k,s+1) & (N-s)\lambda(i,s)\, h + o(h), & s = 0, \dots, N-1, \\ (i,k,s-1) & \min(n,s)(\mu(k,s)/\varepsilon)h + o(h), & s = 1, \dots, N. \end{cases}$$

In addition, the sojourn time $\tau_\varepsilon(i,k,s)$ of $Z_\varepsilon(t)$ in state (i,k,s) is exponentially distributed with parameter

$$a_{ii} + b_{kk} + (N-s)\lambda(i,s) + \min(n,s)\mu(k,s)/\varepsilon.$$

Thus, the transition probabilities for the embedded Markov chain are

$$p_\varepsilon[(i,k,s),(j,k,s)] = \frac{a_{ij}}{a_{ii} + b_{kk} + (N-s)\lambda(i,s) + \min(n,s)\mu(k,s)/\varepsilon},$$

$$p_\varepsilon[(i,k,s),(i,q,s)] = \frac{b_{kq}}{a_{ii} + b_{kk} + (N-s)\lambda(i,s) + \min(n,s)\mu(k,s)/\varepsilon}, \quad \text{for } s = 0,\dots,N,$$

$$p_\varepsilon[(i,k,s),(i,k,s+1)] = \frac{(N-s)\lambda(i,s)}{a_{ii} + b_{kk} + (N-s)\lambda(i,s) + \min(n,s)\mu(k,s)/\varepsilon}, \quad \text{for } s = 0,\dots,N-1,$$

$$p_\varepsilon[(i,k,s),(i,k,s-1)] = \frac{\min(n,s)\mu(k,s)/\varepsilon}{a_{ii} + b_{kk} + (N-s)\lambda(i,s) + \min(n,s)\mu(k,s)/\varepsilon}, \quad \text{for } s = 1,\dots,N.$$

As $\varepsilon \to 0$ this implies

$$\begin{aligned}
p_\varepsilon[(i,k,0),(j,k,0)] &= \frac{a_{ij}}{a_{ii} + b_{kk} + N\lambda(i,0)},\\
p_\varepsilon[(i,k,0),(i,q,0)] &= \frac{b_{kq}}{a_{ii} + b_{kk} + N\lambda(i,0)},\\
p_\varepsilon[(i,k,s),(j,k,s)] &= o(1), \quad s = 1,\dots,N,\\
p_\varepsilon[(i,k,s),(i,q,s)] &= o(1), \quad s = 1,\dots,N,\\
p_\varepsilon[(i,k,0),(i,k,1)] &= \frac{N\lambda(i,0)}{a_{ii} + b_{kk} + N\lambda(i,0)},\\
p_\varepsilon[(i,k,s),(i,k,s+1)] &= \frac{(N-s)\lambda(i,s)\varepsilon}{\min(n,s)\mu(k,s)}(1+o(1)), \quad s = 1,\dots,N-1,\\
p_\varepsilon[(i,k,s),(i,k,s-1)] &\to 1, \quad s = 1,\dots,N.
\end{aligned}$$

This agrees with the conditions 1–4, but here the zero level is the set

$$((i,k,0),(i,k,1),\ i = 1,\dots,r_1,\ k = 1,\dots,r_2)$$

while the q-th level is the set

$$((i,k,q+1),\ i=1,\dots,r_1,\ k=1,\dots,r_2).$$

Since the level 0 in the limit forms an essential class, the probabilities $\pi_0(i,k,0)$, $\pi_0(i,k,1)$, $i=1,\dots,r_1$, $k=1,\dots,r_2$ satisfy the following system of equations

$$\begin{aligned}\pi_0(j,q,0) &= \sum_{i\neq j}\pi_0(i,q,0)a_{ij}/(a_{ii}+b_{qq}+N\lambda(i,0))\\ &+\sum_{k\neq q}\pi_0(j,k,0)b_{kq}/(a_{jj}+b_{kk}+N\lambda(j,0))+\pi_0(j,q,1),\end{aligned}\tag{2}$$

$$\pi_0(j,q,1)=\pi_0(j,q,0)N\lambda(j,0)/(a_{jj}+b_{qq}+N\lambda(j,0)).\tag{3}$$

It is clear that

$$\pi_j^{(1)}a_{jj}=\sum_{i\neq j}\pi_i^{(1)}a_{ij},\quad \pi_q^{(2)}b_{qq}=\sum_{k\neq q}\pi_k^{(2)}b_{kq}.\tag{4}$$

It can easily be verified, that the solution of (2), (3) subject to (4) is

$$\begin{aligned}\pi_0(i,k,0)&=B\pi_i^{(1)}\pi_k^{(2)}(a_{ii}+b_{kk}+N\lambda(i,0)),\\ \pi(i,k,1)&=B\pi_i^{(1)}\pi_k^{(2)}N\lambda(i,0),\end{aligned}$$

where B is the normalizing constant, i.e.

$$1/B=\sum_{i=1}^{r_1}\sum_{k=1}^{r_2}\pi_i^{(1)}\pi_k^{(2)}[a_{ii}+b_{kk}+2N\lambda(i,0)].$$

By using formula (1) it is easy to show that the probability of exit from $\langle\alpha_m\rangle$ is

$$g_\varepsilon(\langle\alpha_m\rangle)=\varepsilon^m NB\sum_{i=1}^{r_1}\sum_{k=1}^{r_2}\pi_i^{(1)}\pi_k^{(2)}\lambda(i,0)\prod_{s=1}^{m}\frac{(N-s)\lambda(i,s)}{\min(n,s)\mu(k,s)}(1+o(1)).$$

Taking into account the exponentiality of $\tau_\varepsilon(j,k,s)$ for fixed θ we have

$$\begin{aligned}E\exp\{i\varepsilon^m\theta\tau_\varepsilon(j,k,0)\}&=1+\varepsilon^m\frac{i\theta}{a_{jj}+b_{kk}+N\lambda(j,0)}(1+o(1)),\\ E\exp\{i\varepsilon^m\theta\tau_\varepsilon(j,k,s)\}&=1+o(\varepsilon^m),\quad s>0.\end{aligned}$$

Notice that $\beta_\varepsilon = \varepsilon^m$ and therefore from Corollary 1 we immediately get the statement that $\varepsilon^m \Omega_\varepsilon(m)$ converges weakly to an exponentially distributed random variable with parameter

$$\Lambda = N \sum_{i=1}^{r_1} \sum_{k=1}^{r_2} \pi_i^{(1)} \pi_k^{(2)} \lambda(i,0) \prod_{s=1}^{m} \frac{(N-s)\lambda(i,s)}{\min(n,s)\mu(k,s)},$$

which completes the proof.

Consequently, the distribution of the time until the number of stopped machines reaches the $(m+1)$-th level for the first time, can be approximated by

$$P(\Omega_\varepsilon(m) > t) = P(\varepsilon^m \Omega_\varepsilon(m) > \varepsilon^m t) \approx \exp(-\varepsilon^m \Lambda t),$$

i.e. $\Omega_\varepsilon(m)$ is asymptotically an exponentially distributed random variable with parameter $\varepsilon^m \Lambda$. In particular, for $m = N-1$, which means that there is no operating machine, we have

$$\begin{aligned} \Lambda^* &= \varepsilon^{N-1}\Lambda = \varepsilon^{N-1} \frac{N!}{n! n^{N-n-1}} \sum_{i=1}^{r_1} \sum_{k=1}^{r_2} \pi_i^{(1)} \pi_k^{(2)} \lambda(i,0) \prod_{s=1}^{N-1} \frac{\lambda(i,s)}{\mu(k,s)} \\ &= \frac{N!}{n! n^{N-n-1}} \sum_{i=1}^{r_1} \sum_{k=1}^{r_2} \pi_i^{(1)} \pi_k^{(2)} \lambda(i,0) \prod_{s=1}^{N-1} \frac{\lambda(i,s)}{\mu(k,s)/\varepsilon}. \end{aligned} \tag{5}$$

In the case when there are no random environments we get

$$\Lambda^* = \frac{N!}{n! n^{N-n-1}} \lambda(0) \prod_{s=1}^{N-1} \frac{\lambda(s)}{\mu(s)/\varepsilon},$$

where

$$\lambda(s) = \lambda(i,s),\ i = 1,\dots,r_1, \quad \mu(s) = \mu(k,s),\ k = 1,\dots,r_2.$$

Finally, for the simplest case we have:

$$\Lambda^* = \frac{N!}{n! n^{N-n-1}} \lambda \Big(\frac{\lambda}{\mu/\varepsilon}\Big)^{N-1}. \tag{6}$$

Hence the steady-state probability Q_W that at least one machine works is

$$Q_W = \frac{\frac{1}{\varepsilon^{N-1}\Lambda}}{\frac{1}{\varepsilon^{N-1}\Lambda} + \sum_{k=1}^{r_2} \pi_k^{(2)} \frac{1}{n\mu(k,N)/\varepsilon}} \tag{7}$$

$$= \frac{1}{1+\frac{N!}{n!n^{N-n}}\left(\sum_{i=1}^{r_1}\sum_{k=1}^{r_2}\pi_i^{(1)}\pi_k^{(2)}\lambda(i,0)\prod_{s=1}^{N-1}\frac{\lambda(i,s)}{\mu(k,s)/\epsilon}\right)\left(\sum_{k=1}^{r_2}\pi_k^{(2)}\frac{1}{\mu(k,N)/\epsilon}\right)}.$$

In the case when there are no random environments we get

$$Q_W = \frac{1}{1+\frac{N!}{n!n^{N-n}}\prod_{s=1}^{N}\frac{\lambda(s-1)}{\mu(s)/\epsilon}}.$$

Finally, for the simplest case we obtain

$$Q_W = \frac{1}{1+\frac{N!}{n!n^{N-n}}\left(\frac{\lambda}{\mu/\epsilon}\right)^N}. \tag{8}$$

4. Some numerical results and applications in textile winding

In the context of the production department on the factory floor, most manufacturers will seek to establish a constant and optimal environment in which the various processes can be carried out. They will try to avoid the random environment. However, we do not live in the ideal world and variations in the repair rate and the breakdown rate will occur in spite of their best efforts. Machine operatives will feel "below par" with physical or mental problems from time to time and this in turn will affect their work rate. Their attitude to work at the start of a shift will be very different from their attitude just prior to the tea-break, just after the tea-break and again before the end of their shift. Of course, one could argue that the latter changes are more deterministic than random, although variations among workers will tend to make the overall effect more random than it might appear to be at first sight. The use of robots, and there is a marked trend in this direction in many industries, seeks to avoid these effects. The machinery used will suffer from minor faults due to wear and tear. These, although they may not in themselves constitute a breakdown, will have an adverse effect on the stoppage rate of the process. Another reason for variability in the stoppage rate arises from the quality of the raw materials used. This material may have been produced at an earlier stage in the production process, and unless very stringent quality control procedures have been used some variability is inevitable. In the particular case of the textile industry, especially where natural fibres such as wool or cotton are being used, variability between batches of raw yarns is difficult to avoid. Although it is not possible to generalise, because of the great variety of industrial production processes which exist, if the unit of time is taken to be the average repair time, then the average run time between successive stoppages due to yarn breaks of a single machine might be anything from about 20 time units to 100 time units. The idea of "fast" repair would therefore seem to be reasonable. However, the factors mentioned earlier could easily cause deviations of the order of 10% to 50% of these times. We do not underestimate the practical difficulties of modelling these features of real manufacturing processes. The random environment idea would seem to be a first step in the right direction.

In this section some numerical examples are given to illustrate the problem in question and the asymptotic results are compared to the classical exact formulae as well as the numerical ones obtained by Gaver et al. [7].

Case 1. In this section we illustrate how "good" the asymptotic results are by comparing them to the exact ones.

$$\text{Here } \rho = \frac{\lambda}{\mu/\varepsilon} \text{ and } P_W = 1 - \frac{N!}{n!n^{N-n}}\rho^N P_0 \quad (\text{ from Palm-formula }).$$

With $n = 3$ using (8) we get the following results (see Table 1).

We can see how Q_W depends on N, ρ and how accurate it is. It should be noted that the greater the N the less the ρ for an acceptable approximation.

Case 2. In this section we show how the system behaves for different values of ρ. For $n = 3$, $\lambda = 1$, $\mu/\varepsilon = 10$ and 20 by using (6), (8) we obtain the following results (see Table 2).

	$\rho = 0.1$		$\rho = 0.05$	
N	Q_W	$1/\Lambda^*$	Q_W	$1/\Lambda^*$
5	0.999977778	1500.00	0.999999305	24000.00
10	0.999999973	1205357.14	1	617142857.00
15	1	81280326.80	1	1.3316 E 12

Table 2

We can observe that in each case Q_W and $1/\Lambda^*$ increase as we expected, but for $\rho = 0.05$ the increase in $1/\Lambda^*$ is very sharp.

Case 3. In this section only the machines operate in a random environment and the failure rates as well as the repair rates do not depend on the number of failed machines. We compare the asymptotic result to the numerical one obtained by Gaver et al. [7] and show how it depends on the intensities of the

	N=5		N=10	
ρ	P_W	Q_W	P_W	Q_W
1	0.936305732	0.310344828	0.950122023	3.60304257 E-3
2^{-1}	0.991023339	0.988558352	0.997252877	0.956928378
2^{-2}	0.999290680	0.999939722	0.999974028	0.999992674
2^{-3}	0.999962376	0.999999686	0.999999921	0.999999999
2^{-4}	0.999998435	0.999999998	1	1
2^{-5}	0.999999943	1		
2^{-6}	0.999999998	1		
2^{-7}	1	1		

	N=15		N=20	
ρ	P_W	Q_W	P_W	Q_W
1	0.950212859	2.43840386 E-6	0.950212932	3.18484253 E-10
2^{-1}	0.997516417	0.324055259	0.997521227	1.99972276 E-3
2^{-2}	0.999991894	0.999989390	0.999993701	0.999920681
2^{-3}	0.999999998	1	1	1
2^{-4}	1	1		

	N=25		N=30	
ρ	P_W	Q_W	P_W	Q_W
1	0.950212932	1.21387279 E-14	0.950212932	1.72490449 E-19
2^{-1}	0.997521248	2.44384274 E-6	0.997521248	1.11126125 E-9
2^{-2}	0.999993850	0.997971651	0.999993856	0.877439658
2^{-3}	1	1	1	1

Table 1

governing Markov chain (see Table 3). Here $\hat{Q}_W = 1 - g_\varepsilon(\langle\alpha_m\rangle)$ and P_W is the steady-state probability that at least one machine works obtained by Gaver et al. [7].

$$N = 5, \quad m = 4, \qquad r_1 = 2, \quad r_2 = 1$$

$$\lambda(1,.) = 0.12 \quad \pi_1^{(1)} = 2/3 \quad \lambda(2,.) = 0.06 \quad \pi_2^{(1)} = 1/3$$

$$\mu(1,.)/\varepsilon = 1.00 \quad \pi_1^{(2)} = 1$$

where $\lambda(i,.) = \lambda(i, s-1)$, $\mu(1,.)/\varepsilon = \mu(1,s)$, $i = 1,2$, $s = 1,\ldots,5$

We can observe that the corresponding probabilities are exact up to almost 3 digits while the mean failure-free operation time is very small compared to Gaver et al. [7] where $1/\Lambda^* \approx 1500$.

a_{11}	a_{22}	Q_W	$1/\Lambda^*$	P_W	$\hat{Q}_W$
50	100	0.99798	494.61	0.99932	0.99997
0.5	1	0.99798	494 61	0.99925	0.99879
0.05	0.1	0.99798	494 61	0.99908	0.99810

Table 3

Case 4. In this section we approximate the results of Gaver et al. [7]) by changing the service rate (see Table 4). The other parameters are the same as Case 3.

$$\mu(1,.)/\varepsilon = 1.00 \quad \pi_1^{(2)} = 1$$

a_{11}	a_{22}	Q_W	$1/\Lambda^*$	P_W	$\hat{Q}_W$
50	100	0.99946	1412.68	0.99932	0.99999
0.5	1	0.99946	1412.68	0.99925	0.99958
0.05	0.1	0.99946	1412.68	0.99908	0.99934

Table 4

This example illustrates that with this service rate we get almost the same results as Gaver et al. [7] but here the formulae are much simpler and we do not need numerical procedures.

Case 5. In this section a general setup is considered. On the one hand we investigate the system's behaviour as a function of the number of repairmen and on the other hand we show how the mean failure-free operation time increases as the failure level increases.

$$N = 5, \quad r_1 = 2, \quad r_2 = 2$$

$$\lambda(1,0) = 0.12,\ \lambda(1,1) = 0.13,\ \lambda(1,2) = 0.14,\ \lambda(1,3) = 0.15,\ \lambda(1,4) = 0.16$$
$$\lambda(2,0) = 0.06,\ \lambda(2,1) = 0.07,\ \lambda(2,2) = 0.08,\ \lambda(2,3) = 0.09,\ \lambda(2,4) = 0.1$$

$$\pi_1^{(1)} = 2/3 \quad \pi_2^{(1)} = 1/3$$

$$\mu(1,1) = 1.35,\ \mu(1,2) = 1.36,\ \mu(1,3) = 1.37,\ \mu(1,4) = 1.38,\ \mu(1,5) = 1.39$$
$$\mu(2,1) = 2.35,\ \mu(2,2) = 2.36,\ \mu(2,3) = 2.37,\ \mu(2,4) = 2.38,\ \mu(2,5) = 2.39$$

$$\pi_1^{(2)} = 1/2 \quad \pi_2^{(2)} = 1/2$$

n	1	2	3	4	5
Q_W	0.999646459	0.999977903	0.999993453	0.999996317	0.999997054
$1/\Lambda^*$	1608.62	12872.71	28964.27	38619.20	38619.20

Table 5

m	1	2	3	4
$1/\Lambda^*$	9.46	76.62	1100.01	28964.27

Table 6

For $m = 4$ the system's characteristic as the function of number of repairmen are in Table 5.

For $n = 3$ the mean failure-free operation time of the system is in Table 6.

We can see that $1/\Lambda^*$ sharply increases as m increases as we expected.

5. Conclusion

In this paper the machine interference problem has been treated supposing that the machines and the repair facility are assumed to operate in independent random environments governed by ergodic Markov chains. The running and repair times of machines are supposed to be exponentially distributed random variables with parameter depending on the number of stopped machines and the state of corresponding varying environment. Assuming that the repair rate is much greater than the failure rate ("fast" service), it is shown that the time until the number of stopped machines reaches a certain level converges weakly, under appropriate norming, to an exponentially distributed random variable. Furthermore, some numerical examples illustrate the problem in question in the field of textile winding.

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