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# ON THE PERIODICITY OF THE RADIX EXPANSION

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Let $\alpha$ be an algebraic integer of degree $n$, let $f(x)$ be the minimal polynomial of $\alpha$, which has $r_1$ real roots and $r_2$ pairs of complex roots. Let $\alpha_1, \dots, \alpha_{r_1}$ be the real roots of $f(x)$, $\alpha_{r_1+1}, \dots, \alpha_{r_1+r_2}$ the nonreal roots of $f(x)$, for which $\alpha_j \neq \overline{\alpha}_k$. We shall consider, as it is usual, the vector space $\mathbf{R_n}$ of sequences of $r_1+r_2$ elements

$$\begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_{r_1+r_2} \end{pmatrix},$$

where $x_1, \dots, x_{r_1}$ are real numbers and $x_{r_1+1}, \dots, x_{r_1+r_2}$ are complex numbers. The dimension of $\mathbf{R_n}$ over the real field is $n$.

We say, that $\alpha$ is a base for the radix expansion of $Z[\alpha]$, if all the elements of $Z[\alpha]$ can be uniquely represented in the form

$$\sum_{k=-K}^{\infty} a_k \alpha_j^k , \quad 0 \leq a_k < N(\alpha).$$

Let $\alpha$ be such a base. In this case for all points of $\mathbf{R_n}$ we have a simultaneous expansion

$$(1) \qquad x_j = \sum_{k=-K}^{\infty} \frac{a_k}{\alpha_j^k}, \quad 1 \leq j \leq r_1 + r_2,$$

where the $a_k$'s are independent of the $j$'s, and $0 \leq a_k < |N(\alpha)|$. We say that this expansion is the radix expansion of the point. The relation (1) is proved in [1]. The first of the authors investigated the existence of the base in [2].

The aim of this paper is to prove the following

**Theorem.** *The radix expansion of an element of $\mathbf{R_n}$ is periodical iff $x_j$ is an element of $Q(\alpha_j)$ and the isomorphism $Q(\alpha_1) \to Q(\alpha_j)$ defined by $\alpha_1 \to \alpha_j$ sends $x_1$ to $x_j$.*

The expansion is said to be periodical, if $a_k = a_{k+l}$, for $k \geq k_0$ with a fixed positive $l$.

One part of the theorem is obvious. If the coefficients are periodical, then we have

$$\sum_{k=k_0}^{\infty} \frac{a_k}{\alpha_j^k} = \sum_{k=k_0}^{k_0+l-1} \frac{a_k}{\alpha_j^k} \frac{1}{1-\frac{1}{\alpha_j^k}},$$

from that it follows, that the $x_j$-s are the conjugates of each others.

If $\alpha$ is a base, then the $|\alpha_j|$'s are greater than one and the series are convergent.

On the other hand let us suppose that the conditions in the theorem are satisfied, then the coordinates of the point have the form

$$\frac{p(\alpha_j)}{c},$$

where $p(x)$ is a polynomial, whose coefficients and the $c$ in the denominator are rational integers. From (1) we have

$$\frac{p(\alpha_j)}{c} = \sum_{k=-K}^{\infty} a_k/\alpha_j^k.$$

Let $r_t^{(j)}$ denote

$$r_t^{(j)} = \alpha_j^t \sum_{k=t+1}^{\infty} a_k/\alpha_j^k.$$

In this case the numbers $cr_t^{(j)}$ are algebraic integers. They are conjugates of each other because of the expression

$$c\alpha_j^t p(\alpha_j) = \sum_{k=-K}^{t} ca_k\alpha_j^{t-k} + cr_t^{(j)}.$$

The following inequalities are valid

$$|cr_t^{(j)}| \leq \frac{c(|N(\alpha)|-1)}{|\alpha_j|-1}. \tag{2}$$

It follows from (2), that the number of the distinct conjugate systems is bounded. There exists a natural number $M$, for which we have for infinitely many $t$ that

$$r_t^{(j)} = r_{t+M}^{(j)}. \tag{3}$$

Let $M$ denote the least positive integer, for which (3) holds for infinitely many $t$. Let $t$ be large enough such that the equality

$$r_s^{(j)} = r_{s+M'}^{(j)}$$

is not valid for all $j$ if $s \geq t$, $0 < M' < M$. If we have

(4) $$r_t^{(j)} = r_{t+M}^{(j)},$$

then we can express the values $r_t^{(j)}$. In this case it is valid that

(5) $$\alpha_j^M r_t^{(j)} = \sum_{k=t+1}^{t+M} a_k \alpha_j^{M+t-k} + r_{t+M}^{(j)}.$$

From (4) and (5) it follows, that

(6) $$r_t^{(j)} = \frac{q_t(\alpha_j)}{\alpha_j^M - 1},$$

where $q_t(x)$ is a polynomial of degree at most $M-1$, and the coefficients of that are the coefficients in the radix expansion.

If it is true that

(7) $$r_t^{(j)} = r_{t+M}^{(j)} = r_{t'}^{(j)} = r_{t'+M}^{(j)},$$

then we have

(8) $$r_{t'}^{(j)} = \frac{q_{t'}(\alpha_j)}{\alpha_j^M - 1} = \frac{q_t(\alpha_j)}{\alpha_j^M - 1}.$$

Because of the unicity of the radix expansion it follows, that

(9) $$q_{t'}(x) = q_t(x)$$

and therefore their coefficients have the same values. We shall prove, that the least integer $t'$, that is greater than $t$ and for which the equality

$$r_{t'}^{(j)} = r_{t'+M}^{(j)}$$

holds, is the integer $t+M$. There exists a number $m$, for which

(10) $$r_{t+M+m}^{(j)} = r_{t+M}^{(j)} = r_t^{(j)}.$$

Let $m$ be the least integer with these properties. Because of the conditions above, we have $m \geq M$. Let us suppose, that $m > M$. From (10) we have

$$r_{t+M}^{(j)} = \frac{s(\alpha_j)}{\alpha_j^m - 1},$$

where $s(x)$ is a polynomial of degree at most $m-1$ with coefficients in the radix expansion. From (10) it follows, that

$$\frac{s(\alpha_j)}{\alpha_j^m - 1} = \frac{q(\alpha_j)}{\alpha_j^M - 1},$$

that is

$$\alpha_j^M s(\alpha_j) + q(\alpha_j) = \alpha_j^m q(\alpha_j) + s(\alpha_j). \tag{11}$$

It follows from the unicity of the radix expansion, that

$$s(x) = x^M s_1(x) + q(x), \tag{12}$$

where $s_1(x)$ is a polynomial of degree at most $m - M - 1$ . From (11) and (12) we have

$$x^M s_1(x) + q(x) = x^{m-M} q(x) + s_1(x),$$

that is

$$\frac{q(\alpha_j)}{\alpha_j^M - 1} = \frac{s_1(\alpha_j)}{\alpha_j^{m-M} - 1}. \tag{13}$$

It is valid that

$$\alpha_j^{m-M} r_{t+M}^{(j)} = (a_{t+M+1}\alpha_j^{m-M-1} + \ldots + a_{t+m}) + r_{t+m}^{(j)}. \tag{14}$$

From (13) and (14) it follows, that

$$\alpha_j^{m-M} r_{t+M}^{(j)} = \alpha_j^{m-M} \frac{s_1(\alpha_j)}{\alpha_j^{m-M} - 1} = s_1(\alpha_j) + \frac{s_1(\alpha_j)}{\alpha_j^{m-M} - 1} = \tag{15}$$

$$= (a_{t+M+1}\alpha_j^{m-M-1} + \ldots + a_{t+m}) + \frac{s_1(\alpha_j)}{\alpha_j^{m-M} - 1}.$$

It follows from (13),(14) and (15), that

$$r^{(j)}_{t+M+(m-M)} = r^{(j)}_{t+m} = \frac{s_1(\alpha_j)}{\alpha_j^{m-M} - 1} = r^{(j)}_{t+M},$$

and in this way we have a contradiction with the minimality of the $m$, so it is true, that

$$m = M.$$

We proved with this line of thought, that

$$r^{(j)}_{t+kM} = r^{(j)}_t$$

and now the periodicity of the coefficients follows from (9).

The same theorem is true, if the $a_k^{(j)}$-s form a total residue system for the $\alpha_j$ in $Z[\alpha_j]$ and the isomorphism $Z[\alpha_1] \to Z[\alpha_j]$ defined by $\alpha_1 \to \alpha_j$ sends $a_k^{(1)}$ to $a_k^{(j)}$ for all $k$.

## References


[1] **Kátai I., Környei I.**, On number systems in algebraic fields (to appear)

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