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AN ANALYSIS OF FUZZY PREFERENCE MODELLING

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Abstract. Our starting point is a fuzzy preference relation and we introduce strict preference, indifference and incomparatibility fuzzy relations. We are looking for such relations that satisfy boundary conditions and certain axioms known in the literature (Ovchinnikov and Roubens [1]). Moreover, they are linear functions of the preference relation. We distinguish the strongly S-complete case from the general case. The general case gives us the strict preference, indifference and incomparatibilty relations proposed by Fodor [3]. We apply our analysis to the dual preference relation. Then we show a heuristic way of establishing the strict preference, indifference and incomparatibility in the strongly S-complete case. This heuristic approach uses the dissimilarity relation. The theory is illustrated by a multi-criteria decision model.

1. Preference modelling

Our starting point is a fuzzy relation R on the set of alternatives. That is, a function

$$R : A \times A \to [0,1]$$

such that for any $a, b \in A$, $R(a,b)$ is the truth value of the statement "a is not worse than b". Then we are to define strict preference, indifference and incomparability as fuzzy relations. We are searching for relations which satisfy certain reasonable conditions (Ovchinnikov and Roubens [1], [2]):

A1. For any two alternatives a, b the values of $P(a,b)$, $I(a,b)$ and $J(a,b)$ depend only on $R(a,b)$ and $R(b,a)$, so there exist functions $p, i, j : [0,1] \times [0,1] \to [0,1]$ such that

$$P(a,b) = p\big(R(a,b),\ R(b,a)\big),$$
$$I(a,b) = i\big(R(a,b),\ R(b,a)\big),$$
$$J(a,b) = j\big(R(a,b),\ R(b,a)\big).$$

A2. $p(x,y)$ is nondecreasing in its first place and nonincreasing in its second place;

$i(x,y)$ is nondecreasing with respect to both arguments;

$j(x, y)$ is nonincreasing with respect to both arguments.

A3. P is antisymmetric, I and J are symmetric relations. A3 asserts that

$$\min\{P(a,b),\ P(b,a)\} = 0,$$
$$I(a,b) = I(b,a) \text{ and}$$
$$J(a,b) = J(b,a) \text{ for any } a,b \in A.$$

Let $T, S : [0,1] \times [0,1] \to [0,1]$ and $n : [0,1] \to [0,1]$ be functions modelling intersection, union and complementation in fuzzy set theory, see Weber [4]. We introduce them following Lukasiewicz (see also Ovchinnikov and Roubens [1]):

$$T(x,y) = \phi^{-1}\big(\max\{\phi(x) + \phi(y) - 1, 0\}\big),$$
$$S(a,b) = \phi^{-1}\big(\min\{\phi(x) + \phi(y), 1\}\big),$$
$$n(x) = \phi^{-1}\big(1 - \phi(x)\big),$$

where ϕ is a strictly increasing continuous function from the unit interval onto itself satisfying boundary conditions $\phi(0) = 0$ and $\phi(1) = 1$.

Fodor [3] proposed the next definitions of strict preference, indifference and incomparability relations:

$$P(a,b) = T\big(R(a,b),\ n[R(b,a)]\big), \tag{1.1}$$
$$I(a,b) = \min\{R(a,b),\ R(b,a)\}, \tag{1.2}$$
$$J(a,b) = \min\{n[R(a,b)],\ n[R(b,a)]\}. \tag{1.3}$$

Now some notions used in this paper are introduced. If R is a fuzzy relation, we denote its dual:

$$R^d(a,b) = n[R(b,a)].$$

A fuzzy relation R is

$$\begin{array}{ll} T-\text{asymmetric if} & T\big(R(a,b),\ R(b,a)\big) = 0, \\ \text{strongly } S-\text{complete if} & S\big(R(a,b),\ R(b,a)\big) = 1 \text{ and} \\ S-\text{transitive if} & S\big(R(a,b),\ R(b,c)\big) \geq R(a,c) \end{array}$$

for any $a, b, c \in A$.

It is easy to check that R is T-asymmetric iff R^d is strongly S-complete. The definitions (1.1) – (1.3) fulfil the conditions A1 – A3. If one supposes that $\phi(x) = x$ then R is strongly S-complete (i.e. $S\,(R(a,b), R(b,a)) = 1$ for any $a, b \in A$) if and only if

$$R(a,b) + R(b,a) \geq 1,$$

or equivalently,

$$1 - R(a,b) + 1 - R(b,a) \leq 1.$$

This inequality implies that

$$\min\{n(R(a,b)),\ n(R(b,a))\} \leq 0.5, \text{ i.e. } J(a,b) \leq 0.5 \text{ for any } a, b \in A.$$

In the next section we modify definitions (1.1) – (1.3) in order to have full range [0,1] for P, I, and J.

2. Analysis

First we obtain boundary conditions for p, i, j.

Lemma. *If R is strongly S-complete and fulfils A1 – A3, then (2.1) – (2.9) hold:*

(2.1) $$p(1,0) = \max_{x,y}\{p(x,y)\} = 1;$$

(2.2) $$p(1,1) = \min_{x,y}\{p(x,y)\} = 0;$$

(2.3) $$p(0.5,0.5) = \min_{x,y}\{p(x,y)\} = 0;$$

(2.4) $$i(1,0) = \min_{x,y}\{i(x,y)\} = 0;$$

(2.5) $$i(1,1) = \max_{x,y}\{i(x,y)\} = 1;$$

(2.6) $$i(0.5,0.5) = \min_{x,y}\{i(x,y)\} = 0;$$

(2.7) $$j(1,0) = \min_{x,y}\{j(x,y)\} = 0;$$

(2.8) $$j(1,1) = \min_{x,y}\{j(x,y)\} = 0;$$

(2.9) $$j(0.5,0.5) = \max_{x,y}\{j(x,y)\} = 1.$$

Proof. The lemma directly follows from the definitions.

We denote $x = R(a,b)$, $y = R(b,a)$ for short. Now we are looking for p, i, j in the next form:

$$\phi(p(x,y)) = p_1^*\phi(x) + p_2^*\phi(y) + p_3,$$
$$\phi(i(x,y)) = i_1^*\phi(x) + i_2^*\phi(y) + i_3,$$
$$\phi(j(x,y)) = j_1^*\phi(x) + j_2^*\phi(y) + j_3,$$

if (x, y) is in the triangle defined by vertices $(1,0)$, $(1,1)$, $(0.5,0.5)$. These functions are completely determined by values in points $(1,0)$, $(1,1)$, $(0.5,0.5)$. Thus our lemma implies the following expressions:

$$\begin{aligned}\phi(p(x,y)) &= \phi(x) - \phi(y),\\ \phi(i(x,y)) &= \phi(x) + \phi(y) - 1,\\ \phi(j(x,y)) &= (1-\phi(x)) \,/\, (1-\phi(0.5)),\end{aligned}$$

or equivalently

$$P(a,b) = T\Big(R(a,b),\ n\big(R(b,a)\big)\Big), \tag{2.10}$$

$$I(a,b) = T\big(R(a,b),\ R(b,a)\big) \tag{2.11}$$

and

$$\begin{aligned} &J(a,b) = \\ &= \min\Big\{\phi^{-1}\Big(\big[1-\phi(R(a,b))\big] \,/\, \big[1-\phi(0.5)\big]\Big),\\ &\phi^{-1}\Big(\big[1-\phi(R(b,a))\big] \,/\, \big[1-\phi(0.5)\big]\Big)\Big\}\end{aligned} \tag{2.12}$$

which holds in the triangle with vertices $(1,0)$, $(1,1)$, $(0,1)$. The formula (2.12) has been proposed by Roubens ([5]), but he obtained it in another way.

Boundary conditions (2.1) – (2.9) can be expressed in the general case, when R is not strongly S-complete and $(x,y) \in [0,1] \times [0,1]$. In this case the values of p, i and j have to be determined in points $(0,0)$, $(0,1)$, $(1,1)$. The relations P, I and J, implied by those boundary conditions, are the same that Fodor proposed, (1.1) – (1.3).

3. Duality

Now the dual model is analysed. Let $Q(a,b)$ be a relation expressing the truth value of the statement "a is better that b". In this case axioms A1 and A2 must be altered:

A1'. For any two alternatives a, b the values of $P(a,b)$, $I(a,b)$ and $J(a,b)$ depend only on $Q(a,b)$ and $Q(b,a)$, so there exist functions $p, i, j : [0,1] \times [0,1] \to [0,1]$ such that

$$\begin{aligned}P(a,b) &= p\big(Q(a,b),\ Q(b,a)\big),\\ I(a,b) &= i\big(Q(a,b),\ Q(b,a)\big),\\ J(a,b) &= j\big(Q(a,b),\ Q(b,a)\big).\end{aligned}$$

A2'. $p(x,y)$ is nondecreasing in its first place and nonincreasing in its second place;

$i(x,y)$ is nonincreasing with respect to both arguments;

$j(x,y)$ is nondecreasing with respect to both arguments.

First the case when Q is T-asymmetric, i.e. $\phi(Q(a,b)) + \phi(Q(b,a)) \leq 1$ for any $a,b \in A$, is analysed. From now we denote $u = Q(a,b)$ and $v = Q(b,a)$. The following boundary conditions can be obtained in the points $(1,0)$, $(0,0)$ and $(0.5,0.5)$:

$$p(1,0) = \max_{u,v\in[0,1]}\{p(u,v)\} = 1$$
$$p(0,0) = \min_{u,v\in[0,1]}\{p(u,v)\} = 0$$
$$p(0.5,0.5) = \min_{u,v\in[0,1]}\{p(u,v)\} = 0$$
$$i(1,0) = \min_{u,v\in[0,1]}\{i(u,v)\} = 0$$
$$i(0,0) = \max_{u,v\in[0,1]}\{i(u,v)\} = 1$$
$$i(0.5,0.5) = \min_{u,v\in[0,1]}\{i(u,v)\} = 0$$
$$j(1,0) = \min_{u,v\in[0,1]}\{j(u,v)\} = 0$$
$$j(0,0) = \min_{u,v\in[0,1]}\{j(u,v)\} = 0$$
$$j(0.5,0.5) = \max_{u,v\in[0,1]}\{j(u,v)\} = 1$$

which implies

$$\phi(p(u,v)) = \phi(u) - \phi(v),$$
$$\phi(i(u,v)) = 1 - \phi(u) - \phi(v) \quad \text{and}$$
$$\phi(j(u,v)) = 2\phi(v).$$

So the relations in the triangle with vertices $(1,0)$, $(0,0)$ and $(0,1)$ are:

(3.1) $$P(a,b) = T\Big(Q(a,b),\ n\big(Q(b,a)\big)\Big),$$

(3.2) $$I(a,b) = n\Big(S\big(Q(a,b),\ Q(b,a)\big)\Big) \text{ and}$$

(3.3) $$J(a,b) = \min\{\phi^{-1}(2Q(a,b)),\ \phi^{-1}(2Q(b,a))\}.$$

The case when $Q(a,b) \in [0,1] \times [0,1]$ provides us

$$P(a,b) = T\Big(Q(a,b),\ n\big(Q(b,a)\big)\Big), \tag{3.4}$$

$$I(a,b) = \min\big\{n\big(Q(a,b)\big),\ n\big(Q(b,a)\big)\big\} \text{ and} \tag{3.5}$$

$$J(a,b) = \min\{Q(a,b),\ Q(b,a)\}. \tag{3.6}$$

However it cannot be said in general that in a certain situation the relation Q representing the notion "a is better than b" could be expressed as the dual of the relation R expressing "a is not worse than b" (because of the incomparability), the formulas derived from the two approaches are duals: if $R(a,b)$ is changed to $n\,(Q(b,a))$ in the formulas (2.10) – (2.12) (or in (1.1) – (1.3)), the expressions (3.1) – (3.3) (or (3.4) – (3.6)) are obtained.

4. A heuristic approach: the dissimilarity relation

In this section we present a heuristic way to derive formulas (2.26) – (2.27).

In mathematical statistics similarity S and dissimilarity D relations are used to express the connection between two objects (see Hardine and Sibson [6]). Concerning to the notions of fuzzy theory (Höhle [7]), similarity and dissimilarity relations are symmetric fuzzy relations on A and the equality

$$D(a,b) = n\big(S(a,b)\big) \quad \text{or any} \quad a,b \in A \tag{4.1}$$

expresses their basic connection. The condition

$$D(a,b) = 0 \quad \text{if and only if} \quad a = b \tag{4.2}$$

is supposed in certain cases. If D is S-transitive and fulfils (4.2), then D is a norm on A (S-transitivity implies the triangle inequality).

The notion of indifference can be seen as a similarity of two alternatives. We will follow this approach.

For descriptiveness an example is presented. Let A be the set of the alternatives, $c_1,\ c_2,\dots,c_k$ criteria with $w_1,\ w_2,\dots,w_k$, representing their relative importances $\big(\sum w_i = 1\big)$, and $>_1,\ >_2,\dots,>_k;\ =_1,\ =_2,\dots,=_k$ complete orders on A. We will use $\phi(x) = x$ for short. Let

$$w^+(a,b) = \sum_{i,a>_i b} w_i,$$

$$w^=(a,b) = \sum_{i,a=_i b} w_i \quad \text{and}$$

$$w^-(a,b) = \sum_{i,a<_i b} w_i.$$

It is clear in this case that

$$R(a,b) = w^+(a,b) + w^=(a,b) \text{ and}$$
$$Q(a,b) = w^+(a,b).$$

Let the dissimilarity be a fuzzy relation on A expressing the truth value of the statement "a differs from b" for any a, b alternatives. In the example this notion can be expressed as:

$$D(a,b) = w^+(a,b) + w^-(a,b)$$

and a possible definition is:

$$D(a,b) = S\big(Q(a,b),\ Q(b,a)\big) \text{ or} \tag{4.3}$$

$$D(a,b) = n\Big(T\big(R(a,b), R(b,a)\big)\Big). \tag{4.4}$$

Obviously D is symmetric. If Q is S-transitive (as in the example), then D is S-transitive, too. Moreover, if

$$Q(a,b) = 0 \quad \text{if and only if} \quad a = b \quad (D \text{ fulfils } (4.2)),$$

then D is a norm on A.

Now the indifference and incomparability is expressed by the dissimilarity and preference. The indifference can be interpreted as $n\,(D(a,b))$ (concerning to (4.1)) and (3.2) is obtained immediately. If $P(a,b)$, $P(b,a)$ are small and $D(a,b)$ is big, a and b must be incomparable:

$$J(a,b) = T\Big(n\big[S\big(P(a,b),\ P(b,a)\big)\big],\ D(a,b)\Big),$$

which is equivalent to (2.27) form.
Obviously R is strongly S-complete and Q is T-asymmetric. Using (2.10) – (2.12)

$$P(a,b) = w^+(a,b) - w^-(a,b),$$
$$I(a,b) = w^=(a,b) \text{ and}$$
$$J(a,b) = 2w^-(a,b)$$

can be obtained if $w^+(a,b) \geq w^-(a,b)$.

5. Conclusion

We established strict preference, indifference and incomparability relations which satisfy certain axioms and boundary conditions. If the preference relation is strongly S-complete, we used special boundary conditions so that the ranges of the relations would be the whole unit interval. In the general case, our approach resulted in relations proposed by Fodor.

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