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NONLINEAR ELLIPTIC EQUATIONS WITH NONLINEAR INTEGRAL CONDITION ON THE BOUNDARY

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The aim of this paper is to prove existence of solutions of second order partial differential equations in a domain $\Omega \subset \mathbb{R}^n$ with the following nonlocal boundary conditions:

$$u(x) = h_1\Big(x, u(\Phi(x))\Big) + \int\limits_{\partial\Omega} h_2\Big(x, t, u(\Psi(t))\Big) d\sigma_t \tag{0.1}$$

resp.

$$\partial_{\nu^*} u := h_1(x, u(x)) + h_2\Big(x, u(\Phi(x))\Big) + \int\limits_{\partial\Omega} h_3\Big(x, t, u(\Psi(t))\Big) d\sigma_t; \tag{0.2}$$

$x \in \partial\Omega$,

where $\partial_{\nu^*} u$ denotes the "conormal derivative" of u; Φ, Ψ are given continuous mappings from $\partial\Omega$ into $\overline{\Omega}$.

Linear elliptic equations with nonlocal boundary condition have been considered firtsly in [4] and they by several authors (see e.g. [3], [5], [14], [15] and [16]). Nonlinear elliptic equations with nonlocal boundary condition have been studied in [11] and [12]. Similar problems with nonlocal boundary condition, without integral term, have been considered in [7] and [8].

In [6] it is proved the following comparison principle. Let Q be a second order quasilinear elliptic operator defined by the formula

$$Q(u) := \sum_{i,j=1}^{n} a_{ij}(x, u, \partial u)\partial_i\partial_j u + b(x, u, \partial u)$$

where $x = (x_1, x_2, \ldots, x_n) \in \Omega \subset \mathbb{R}^n$, $n \geq 2$ and $u \in C^2(\Omega)$. The coefficients $a_{ij}(x, z, p)$, $(i, j = 1, \ldots, n)$, $b(x, z, p)$ are assumed to be real valued and defined for all values of (x, z, p) in $\Omega \times \mathbb{R} \times \mathbb{R}^n$, further $a_{ij} = a_{ji}$, Ω is bounded.

Theorem A. *Let $u, v \in C(\overline{\Omega}) \cap C^2(\Omega)$ satisfy $Q(u) \geq Q(v)$ in Ω, $u \leq v$ on $\partial\Omega$, where*

(i) *the operator Q is elliptic;*
(ii) *the coefficients $a_{ij}(x, z, p)$ are independent of z;*
(iii) *the coefficient $b(x, z, p)$ is nonincreasing in z for each $(x, p) \in \Omega \subset \mathbb{R}^n$;*
(iv) *the coefficients a_{ij}, b are continuously differentiable in $\Omega \times \mathbb{R} \times \mathbb{R}^n$.*

Then $u \leq v$ in Ω.

In [6] there are formulated conditions such that the Dirichlet problem

$$(0.3) \qquad \begin{aligned} Q(u) &= 0 \quad \text{in} \quad \Omega, \\ u &= \varphi \quad \text{on} \quad \partial\Omega \end{aligned}$$

has a solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$ for any $\varphi \in C(\partial\Omega)$ (see Theorem 15.18 of [6]).

1. First boundary value problem

Consider the following problem

$$(1.1) \qquad Q(u) := \sum_{i,j=1}^{n} a_{ij}(x, u, \partial u)\partial_i\partial_j u + b(x, u, \partial u) = 0 \quad \text{in} \quad \Omega,$$

$$(1.2) \qquad u(x) = h_1\Big(x, u\big(\Phi(x)\big)\Big) + \int_{\partial\Omega} h_2\Big(x, t, u\big(\Psi(t)\big)\Big) d\sigma_t \quad \text{on} \quad \partial\Omega,$$

where $\Phi, \Psi : \partial\Omega \to \overline{\Omega}$ are continuous mappings, and $h_1 : \partial\Omega \times \mathbb{R} \to \mathbb{R}$, $h_2 : \partial\Omega \times \mathbb{R}^2 \to \mathbb{R}$ are continuous functions such that $|\partial_2 h_1|$, $|\partial_3 h_2|$ exist with the property $\big(\sup|\partial_2 h_1| + \lambda(\partial\Omega)\sup|\partial_3 h_2|\big) < 1$, $\lambda(\partial\Omega)$ is the measure of surface $\partial\Omega$.

We shall prove existence and uniqueness of the solution of problem (1.1), (1.2) by using arguments of [8]. The main result of this paragraph is the following

Theorem 1. *Assume that the above conditions and conditions (i) - (iv) of Theorem A are fulfilled with hypothesis of Theorem 15.18 of [6]. Then there exists a unique solution of (1.1), (1.2).*

Proof. Denote by $G(\varphi)$ the solution u of the Dirichlet problem (0.3). Further define operator B by

$$B(\varphi)(x) := h_1\Big(x, G(\varphi)\big(\Phi(x)\big)\Big) + \int_{\partial\Omega} h_2\Big(x, t, G(\varphi)\big(\Psi(t)\big)\Big) d\sigma_t,$$

then $B : C(\partial\Omega) \to C(\partial\Omega)$ is a nonlinear mapping, where $C(\partial\Omega)$ is a complete metric space with the metric $\rho(\varphi_1, \varphi_2) := \sup|\varphi_1 - \varphi_2|$.

It is easy to prove that if $\varphi \in C(\partial\Omega)$ is a fixed point of B, i.e. $B(\varphi) = \varphi$, then $u := G(\varphi)$ is a solution of (1.1), (1.2) and, conversely, if u is a solution of (1.1), (1.2), then $\varphi := u|_{\partial\Omega}$ is a fixed point of B.

Therefore to prove the existence of (1.1), (1.2) it is sufficient to show that B has a fixed point. This will be a consequence of Banach's fixed point theorem.

Now we show that $B : C(\partial\Omega) \to C(\partial\Omega)$ is a contraction on $C(\partial\Omega)$ for any $\varphi_1, \varphi_2 \in C(\partial\Omega)$

$$\rho\big(B(\varphi_1), B(\varphi_2)\big) = \sup|B(\varphi_1) - B(\varphi_2)| \le q \cdot \rho(\varphi_1, \varphi_2), \tag{1.3}$$

where $q := \big(\sup|\partial_2 h_1| + \lambda(\partial\Omega) \cdot \sup|\partial_3 h_2|\big) < 1$. We have

$$[B(\varphi_1)](x) - [B(\varphi_2)](x) =$$

$$= \left\{ h_1\big[x, G(\varphi_1)(\Phi(x))\big] + \int\limits_{\partial\Omega} h_2\big[x, t, G(\varphi_1)(\Psi(t))\big] d\sigma_t \right\} -$$

$$- \left\{ h_1\big[x, G(\varphi_2)(\Phi(x))\big] - \int\limits_{\partial\Omega} h_2\big[x, t, G(\varphi_2)(\Psi(t))\big] d\sigma_t \right\}.$$

Further, by using Lagrange's mean value theorem and the notations

$$a_j := G(\varphi_j)\big(\Phi(x)\big), \quad b_j := G(\varphi_j)\big(\Psi(t)\big), \qquad (j = 1, 2)$$

we find that

$$[B(\varphi_1)](x) - [B(\varphi_2)](x) = \partial_2 h_1(x, a_2 + c[a_1 - a_2])(a_1 - a_2) +$$

$$+ \int\limits_{\partial\Omega} \partial_3 h_2(x, t, b_2 + \tilde{c}[b_1 - b_2])(b_1 - b_2) d\sigma_t.$$

Consequently,

$$|B(\varphi_1)(x) - B(\varphi_2)(x)| \le \sup|\partial_2 h_1| \big|G(\varphi_1)\big(\Phi(x)\big) - G(\varphi_2)\big(\Phi(x)\big)\big| +$$

$$+ \sup|\partial_3 h_2| \cdot \int\limits_{\partial\Omega} \big|G(\varphi_1)\big(\Psi(t)\big) - G(\varphi_2)\big(\Psi(t)\big)\big| d\sigma_t.$$

We shall prove that

$$(1.4)\qquad \begin{aligned} \big|G(\varphi_1)(\Phi(x)) - G(\varphi_2)(\Phi(x))\big| &\le \rho(\varphi_1,\varphi_2); \\ \big|G(\varphi_1)(\Psi(t)) - G(\varphi_2)(\Psi(t))\big| &\le \rho(\varphi_1,\varphi_2). \end{aligned}$$

From these inequalities it follows

$$\rho\big(B(\varphi_1), B(\varphi_2)\big) \le q \cdot \rho(\varphi_1,\varphi_2),$$

where $q := \big(\sup|\partial_2 h_1| + \lambda(\partial\Omega)\cdot \sup|\partial_3 h_2|\big)$. This means that B is a contraction in $C(\partial\Omega)$. By using conditions of theorem A we want to prove that for all $y := \Phi(x) \in \overline{\Omega}$

$$|G(\varphi_1)(y) - G(\varphi_2)(y)| \le \sup_{\partial\Omega} |\varphi_1 - \varphi_2|.$$

Let $u_1 := G(\varphi_1),\ u_2 := G(\varphi_2)$, then we have

$$Q(u_1) = Q(u_2) = 0 \ \text{ in } \ \Omega, \quad u_1 = \varphi_1, \quad u_2 = \varphi_2 \ \text{ on } \ \partial\Omega.$$

We shall show that this implies

$$|u_1(y) - u_2(y)| \le \sup_{\partial\Omega} |\varphi_1 - \varphi_2| \quad \text{for all} \quad y \in \Omega.$$

By using notation $\varepsilon := \sup_{\partial\Omega} |\varphi_1 - \varphi_2|$ we may write $\varphi_1 - \varepsilon \le \varphi_2 \le \varphi_1 + \varepsilon$. Consider the functions $u := u_2,\ v := u_1 + \varepsilon$. Since

$$Q(u_1+\varepsilon) = \sum_{i,j=1}^{n} a_{ij}\big(x, \partial(u_1+\varepsilon)\big)\partial_i(u_1+\varepsilon)\partial_j(u_1+\varepsilon) +$$

$$+ b\big(x, u_1+\varepsilon, \partial(u_1+\varepsilon)\big) \le \sum_{i,j=1}^{n} a_{ij}(x, \partial u_1)(\partial_i u_1)(\partial_j u_1) +$$

$$+ b(x, u_1, \partial u_1) = Q(u_1) = 0,$$

thus

$$Q(v) = Q(u_1+\varepsilon) \le 0 = Q(u_2) = Q(u) \quad \text{in} \quad \Omega.$$

Further,

$$v = u_1 + \varepsilon = \varphi_1 + \varepsilon \ge \varphi_2 = u_2 = u \quad \text{on} \quad \partial\Omega.$$

It means that all conditions of Theorem A are fulfilled, thus $u \le v$ in Ω, i.e. for all $y \in \Omega$

$$u_2(y) \le u_1(y) + \varepsilon.$$

Similarly can be proved that for all $y \in \Omega$

$$u_1(y) - \varepsilon \le u_2(y)$$

and so we have

$$|u_1(y) - u_2(y)| \le \varepsilon.$$

Thus we have shown that

$$\big|G(\varphi_1)(\Phi(x)) - G(\varphi_2)(\Phi(x))\big| \le \sup|\varphi_1 - \varphi_2| = \rho(\varphi_1, \varphi_2);$$
$$\big|G(\varphi_1)(\Psi(t)) - G(\varphi_2)(\Psi(t))\big| \le \sup|\varphi_1 - \varphi_2| = \rho(\varphi_1, \varphi_2).$$

Hence we obtain (1.3) which completes the proof of Theorem 1.

Since the operator B has exactly one fixed point thus the solution of (1.1), (1.2) is unique.

Theorem 2. *Assume that Q satisfies the conditions of Theorem 15.18 of [6] and $\Phi, \Psi : \partial\Omega \to \partial\Omega$ are continuous mappings, h_1, h_2 satisfy the same conditions as in Theorem 1, then there exists a unique solution of (1.1), (1.2).*

The proof of Theorem 2 is similar to the proof of Theorem 1 except of the proof of (1.4). Since $\Phi : \partial\Omega \to \partial\Omega$, $\Psi : \partial\Omega \to \partial\Omega$, thus for $x \in \partial\Omega$ we have

$$G(\varphi_1)(\Phi(x)) = \varphi_1(\Phi(x)), \quad G(\varphi_2)(\Phi(x)) = \varphi_2(\Phi(x)),$$

and

$$G(\varphi_1)(\Psi(t)) = \varphi_1(\Psi(t)), \quad G(\varphi_2)(\Psi(t)) = \varphi_2(\Psi(t))$$

and so (1.4) is trivially valid.

Remark 1. If the condition

$$\big(\sup|\partial_2 h_1| + \lambda(\partial\Omega)\sup|\partial_3 h_2|\big) < 1$$

is not fulfilled then the nonlocal boundary value problem may have no solution or it may have several solutions (see [8]).

2. Third boundary value problem

Consider the following problem:

$$\sum_{|\alpha|\le 1} (-1)^{|\alpha|}\partial^\alpha f_\alpha(x, u, \partial_1 u, \dots, \partial_n u) = F \text{ in } \Omega, \tag{2.1}$$

$$\partial_{\nu^*} u = h_1(x, u(x)) + h_2\Big(x, u(\Phi(x))\Big) + \int\limits_{\partial\Omega} h_3\Big(x, t, u(\Psi(t))\Big) d\sigma_t \tag{2.2}$$

on $\partial\Omega$,

where $\partial_{\nu^*} u := \sum\limits_{|\alpha|=1} [f_\alpha(x, u, \partial_1 u, \ldots, \partial_n u)]\nu_\alpha$, ν_α denote the coordinates of the normal unit vector on $\partial\Omega$; Φ, Ψ are C^1-diffeomorphisms in a neighbourhood of $\partial\Omega$ such that $S := \Phi(\partial\Omega) \subset \overline{\Omega}$, $\Gamma := \Psi(\partial\Omega) \subset \overline{\Omega}$, $\partial\Omega$ is bounded and continuously differentiable (Ω may be unbounded).

It will be proved the existence of weak solution of (2.1), (2.2) by using arguments of [10], [13].

The weak solution of (2.1), (2.2) will be defined as follows. Assume that u is a classical solution of (2.1), (2.2). Consider any $v \in C^1(\overline{\Omega})$ with bounded support, multiply the differential equation (2.1) by v, by using integral transformations, and by the Gauss-Ostrogradsky theorem we obtain

$$\sum_{|\alpha|\leq 1} \int\limits_{\Omega} [f_\alpha(x, u, \partial_1 u, \ldots, \partial_n u)]\partial^\alpha v - \int\limits_{\partial\Omega} h_1(x, u(x)) v(x) d\sigma_x -$$

$$- \int\limits_{S} \tilde{h}_2(x, u(x)) v(\Phi^{-1}(x)) d\sigma_x - \tag{2.3}$$

$$- \int\limits_{\partial\Omega} \left\{ \int\limits_{\Gamma} \tilde{h}_3(x, \tau, u(\tau)) d\sigma_\tau \right\} v(x) d\sigma_x = \int\limits_{\Omega} Fv =: \langle \tilde{F}, v\rangle.$$

Thus the weak solution of (2.1), (2.2) will be defined by (2.3).

3. Existence theorem

Denote by $W_p^1(\Omega)$ the Sobolev space of real valued functions u, whose distributional derivatives of order ≤ 1 belong to $L^P(\Omega)$ $(1 < p < \infty)$. The norm in $W_p^1(\Omega)$ is defined by

$$\|u\|_{W_p^1(\Omega)} := \left\{ \sum_{|\alpha|\leq 1} \int\limits_{\Omega} |\partial^\alpha u|^p \right\}^{1/p}$$

The points $\xi \in \mathbb{R}^{n+1}$ will be written also in the form $\xi = (\eta, \zeta)$ where $\eta \in \mathbb{R}$, and $\zeta \in \mathbb{R}^n$.

Assume that

a) Functions $f_\alpha, h_1, \tilde{h}_2$ and $\tilde{h}_3$ satisfy the Carathéodory conditions, i.e. they are measurable in x for each ξ resp. η and continuous in ξ resp. η for a.e. $x \in \Omega$.

b) There exist constants $c_1 > 0$, p $(1 < p < \infty)$, and a function $k_1 \in L^q(\Omega)$, where $\frac{1}{p} + \frac{1}{q} = 1$ such that

$$|f_\alpha(x,\xi)| \le c_1|\xi|^{p-1} + k_1(x) \quad \text{for all} \quad \xi \in \mathbb{R}^{n+1}, \quad \text{a.e.} \quad x \in \Omega.$$

c) For all (η,ζ), $(\eta,\zeta') \in \mathbb{R}^{n+1}$ with $\zeta \ne \zeta'$ and a.e. $x \in \Omega$

$$\sum_{|\alpha|=1} [f_\alpha(x,\eta,\zeta) - f_\alpha(x,\eta,\zeta'](\xi_\alpha - \xi'_\alpha) > 0.$$

d) There exist a constant $c_2 > 0$ and a function $k_2 \in L^1(\Omega)$ such that for a.e. $x \in \Omega$ and all $\xi \in \mathbb{R}^{n+1}$

$$\sum_{|\alpha|\le 1} f_\alpha(x,\xi)\xi_\alpha \ge c_2|\xi|^p - k_2(x).$$

e) If $n \ge p$ then there exist constants ρ_1, $\tilde{c}_1 > 0$ and a fixed function $\tilde{k}_1 \in$ $\in L^{1+1/\rho_1}(\partial\Omega)$ such that for all $\eta \in \mathbb{R}$, a.e. $x \in \partial\Omega$

$$|h_1(x,\eta)| \le \tilde{c}_1|\eta|^{\rho_1} + \tilde{k}_1(x),$$

where

$$0 < \rho_1 < \frac{n(p-1)}{n-p} \quad \text{if} \quad n > p$$

$$0 < \rho_1 < \infty \quad \text{if} \quad n = p.$$

If $n < p$ then for any number $s > 0$ there is a function $h_{1,s} \in L^1(\partial\Omega)$ such that

$$|h_1(x,\eta)| \le h_{1,s}(x) \quad \text{if} \quad |\eta| \le s.$$

f) For any $\eta \in \mathbb{R}$, a.e. $x \in \partial\Omega$ we have

$$h_1(x,\eta)\eta \le 0.$$

g) There exist contants $\tilde{c}_2 > 0$, ρ_2 and a fixed function $\tilde{k}_2 \in L^{1+1/\rho_2}(S)$ such that for any $\eta \in \mathbb{R}$, $x \in S$

$$|\tilde{h}_2(x,\eta)| \leq \tilde{c}_2|\eta|^{\rho_2} + \tilde{k}_2(x), \quad 0 < \rho_2 < p-1.$$

i) There exist $c_3 > 0$, ρ_3 and a fixed function $k_3 \in L^{1+1/\rho_3}(\Gamma)$ such that for a.e. $x \in \partial\Omega$, all $\eta \in \mathbb{R}$, $\tau \in \Gamma$

$$|\tilde{h}_3(x,\tau,\eta)| \leq c_3|\eta|^{\rho_3} + k_3(\tau), \quad \text{where} \quad 0 < \rho_3 < p-1.$$

Theorem 3. *Assume that conditions a) – i) are fulfilled. Then for any* $\tilde{F} \in (W_p^1(\Omega))'$ *there exists* $u \in W_p^1(\Omega)$ *which satisfies (2.3) for all* $v \in W_p^1(\Omega)$ *with compact support.*

To the proof of Theorem 3 we shall prove two lemmas. For arbitrary $u, v \in W_p^1(\Omega)$ define

$$\langle A_0(u), v\rangle := \int_\Omega f_\alpha(x,u,\ldots,\partial_n u)\partial^\alpha v,$$

$$\langle B_1(u), v\rangle := \int_{\partial\Omega} h_1\big(x,u(x)\big)v(x)d\sigma_x,$$

$$\langle B_2(u), v\rangle := \int_S \tilde{h}_2\big(x,u(x)\big)v\big(\Phi^{-1}(x)\big)d\sigma_x,$$

$$\langle B_3(u), v\rangle := \int_{\partial\Omega}\left\{\int_\Gamma \tilde{h}_3\big(x,\tau,u(\tau)\big)d\sigma_\tau\right\}v(x)d\sigma_x$$

and

$$A := A_0 - B_1 - B_2 - B_3.$$

Lemma 1. *The operator*

$$A : W_p^1(\Omega) \to \big(W_p^1(\Omega)\big)'$$

is (bounded and) pseudomonotone.

Proof. Firstly we shall prove that A is a bounded operator. A_0, B_1 and B_2 are bounded (see [7]).

Similarly to operators B_1, B_2, the boundedness of B_3 can be proved as follows. We know that the trace operator

$$W_p^1(\Omega) \to L^{\tilde{q}}(\partial\Omega)$$

is compact (and so bounded) if

$$1 \leq \tilde{q} < \frac{(n-1)p}{n-p} \quad \text{for} \quad n > p,$$

$$1 \leq \tilde{q} < \infty \quad \text{for} \quad n = p,$$

and

$$1 \leq \tilde{q} \leq \infty \quad \text{for} \quad n < p.$$

From condition i) we obtain

$$|\langle B_3(u), v\rangle| = \left| \int_{\partial\Omega} \left\{ \int_{\Gamma} \tilde{h}_3(x, \tau, u(\tau)) d\sigma_\tau \right\} v(x) d\sigma_x \right| \leq$$

$$\leq \int_{\partial\Omega} \left| \int_{\Gamma} \tilde{h}_3(x, \tau, u(\tau)) d\sigma_\tau \right| |v(x)| d\sigma_x \leq$$

$$\leq \int_{\partial\Omega} \left[\int_{\Gamma} \left|\tilde{h}_3(x, \tau, u(\tau))\right| d\sigma_\tau \right] |v(x)| d\sigma_x \leq$$

$$\leq \left\{ \int_{\Gamma} [c_3 |u(\tau)|^{\rho_3} + k_3(\tau)] \, d\sigma_\tau \right\} \int_{\partial\Omega} |v(x)| d\sigma_x \leq$$

$$\leq \text{const} \cdot \left\{ \int_{\Gamma} [c_3 |u(\tau)|^{\rho_3} + k_3(\tau)] \, d\sigma_\tau \right\} \|v\|_{W_p^1(\Omega)} \leq$$

$$\leq \text{const} \cdot \left\{ \|u\|^{\rho_3}_{W_p^1(\Omega)} + \int_{\Gamma} k_3(\tau) d\sigma_\tau \right\} \|v\|_{W_p^1(\Omega)},$$

where $\rho_3 < p < \dfrac{(n-1)p}{n-p}$, and thus the trace operator $W_p^1(\Omega) \to L^{\rho_3}(\partial\Omega)$ is bounded. The above estimation implies that $B_3 : W_p^1(\Omega) \to \left(W_p^1(\Omega)\right)'$ is bounded.

From conditions b), c) and Carathéodory conditions it follows that A_0 is pseudomonotone operator (see [2]). Let (u_j) be a sequence such that (u_j) converges weakly in $W_p^1(\Omega)$ to u and

$$\lim_{j\to\infty} \sup\langle A(u_j), u_j - u\rangle \le 0.$$

Firstly we shall prove that

$$\lim_{j\to\infty} \langle B_k(u_j), u_j - u\rangle = 0, \quad (k = 1, 2, 3). \tag{2.4}$$

For $k = 1, 2$ (2.4) was proved in [7]. Case $k = 3$ can be considered in a similar way. We know (by compact the imbedding theorem) that if (u_j) converges weakly to u in $W_p^1(\Omega)$ then there exists a subsequence $(\tilde{u}_j)$ of (u_j) such that $\tilde{u}_j|_{\partial\Omega}$ converges to u in $L^{\tilde{q}}(\partial\Omega)$, where $\tilde{q} := \rho_3 + 1 < p$. By using Hölder's inequality $\left(\text{with } \frac{1}{\tilde{p}} + \frac{1}{\tilde{q}} = 1\right)$, condition i) and the boundedness of the trace operator we have

$$|\langle B_3(\tilde{u}_j), \tilde{u}_j - u\rangle| = \left| \int_{\partial\Omega} \left\{ \int_{\Gamma} \tilde{h}_3(x, \tau, \tilde{u}_j(\tau)) d\sigma_\tau \right\} (\tilde{u}_j - u) d\sigma_x \right| \le$$

$$\le \int_{\partial\Omega} \left| \int_{\Gamma} \tilde{h}_3(x, \tau, \tilde{u}_j(\tau)) d\sigma_\tau \right| |\tilde{u}_j - u| d\sigma_x \le$$

$$\le \left\{ \int_{\partial\Omega} \left[\int_{\Gamma} |\tilde{h}_3(x, \tau, \tilde{u}_j(\tau)) d\sigma_\tau| \right]^{\tilde{p}} \right\}^{1/\tilde{p}} \left\{ \int_{\partial\Omega} |\tilde{u}_j(x) - u(x)|^{\tilde{q}} \right\}^{1/\tilde{q}} \le$$

$$\le \text{const} \cdot \left\{ \int_{\Gamma} |(c_3|u(\tau)|^{\rho_3} + k_3(\tau))|^{\tilde{p}} \right\}^{1/\tilde{p}} \cdot ||\tilde{u}_j - u||_{L^{\tilde{q}}(\partial\Omega)} = \tag{2.5}$$

$$= \text{const} \cdot \left\{ \int_{\Gamma} (c_3|u(\tau)|^{\rho_3} + k_3(\tau))^{(\rho_3+1)/\rho_3} d\sigma_\tau \right\}^{\rho_3/(\rho_3+1)} \cdot ||\tilde{u}_j - u||_{L^{\tilde{q}}(\partial\Omega)} \le$$

$$\leq \text{const} \cdot \left\{ \left[\int_\Gamma |u|^{\rho_3+1} d\sigma \right]^{\rho_3/(\rho_3+1)} + \|k_3\|_{L^{1+1/\rho_3}(\Gamma)} \right\} \cdot \|\tilde{u}_j - u\|_{L^{\tilde{q}}(\partial\Omega)} \leq$$

$$\leq \text{const} \cdot \left\{ \|u\|^{\rho_3}_{W^1_p(\Omega)} + c \right\} \cdot \|\tilde{u}_j - u\|_{L^{\tilde{q}}(\partial\Omega)}.$$

In the last product the first term is bounded and the second term tends to 0. Consequently, (2.4) is proved for a subsequence, $k = 3$ and it is not difficult to show that (2.4) is true also for the original sequence (u_j).

Further, we shall prove that

$$\begin{aligned} B_k(u_j) &\overset{w'}{\to} B_k(u) \quad \text{in} \quad \left(W^1_p(\Omega)\right)', \quad k = 1, 2, 3, \\ \langle B_k(u_j), v\rangle &\to \langle B_k(u), v\rangle. \end{aligned} \tag{2.6}$$

For $k = 1, 2$ see [7]. Now we shall prove (2.6) for $k = 3$, similarly to the case $k = 1, 2$. We have seen that there exists a subsequence $(\tilde{u}_j)$ of (u_j) such that $\tilde{u}_j|_{\partial\Omega}$ converges to u in $L^{\tilde{q}}(\partial\Omega)$. Thus it may be supposed that $(\tilde{u}_j)$ converges a.e. to u on $\partial\Omega$. Consequently, by a)

$$\tilde{h}_3(x, \tau, \tilde{u}_j(\tau)) \to \tilde{h}_3(x, \tau, u(\tau)) \quad \text{a.e. on} \quad \partial\Omega.$$

Now we shall we use Vitali's convergence theorem. By Hölder's inequality and the boundedness of the trace operator, we have

$$\left| \int_E \left\{ \int_\Gamma \tilde{h}_3(x, \tau, \tilde{u}_j(\tau)) d\sigma_\tau \right\} v(x) d\sigma_x \right| \leq$$

$$\leq \left\{ \int_E \left| \int_\Gamma \tilde{h}_3(x, \tau, \tilde{u}_j(\tau)) d\sigma_\tau \right|^{\tilde{p}} d\sigma_x \right\}^{1/\tilde{p}} \cdot \left\{ \int_E |v(x)|^{\tilde{q}} d\sigma_x \right\}^{1/\tilde{q}} \leq$$

$$\leq \left\{ \int_{\partial\Omega} \left| \int_\Gamma \tilde{h}_3(x, \tau, \tilde{u}_j(\tau)) d\sigma_\tau \right|^{\tilde{p}} d\sigma \right\}^{1/\tilde{p}} \cdot \left\{ \int_E |v(x)|^{\tilde{q}} d\sigma_x \right\}^{1/\tilde{q}} < c \cdot \varepsilon$$

if the measure of E is sufficiently small, since $\int\limits_{\partial\Omega} \left| \int\limits_{\Gamma} \tilde{h}_3(x,\tau,\tilde{u}_j(\tau))d\sigma_\tau \right|^{\tilde{p}} d\sigma < c$ (see (2.5)). So it is not difficult to show that all conditions of Vitali's theorem are fulfilled and thus we obtain

$$\lim_{j\to\infty} \langle B_3(\tilde{u}_j), v\rangle = \langle B_3(u), v\rangle$$

for all $v \in W_p^1(\Omega)$. It is easy to prove that the above equality is true also for the original sequence (u_j) and so we have (2.6).

We have shown that if (u_j) converges weakly to u in $W_p^1(\Omega)$ and

$$\lim_{j\to\infty} \langle A(u_j), u_j - u\rangle \leq 0$$

then

$$\lim_{j\to\infty} \langle B_k(u_j), u_j - u\rangle = 0, \quad k = 1,2,3 \tag{2.7}$$

and

$$B_k(u_j) \xrightarrow{w'} B_k(u) \quad \text{in} \quad \left(W_p^1(\Omega)\right)'. \tag{2.8}$$

From (2.7) it follows that

$$\lim_{j\to\infty} \sup \langle A_o(u_j), u_j - u\rangle \leq 0.$$

Since A_0 is pseudomonotone thus $(A_0(u_j)) \xrightarrow{w'} A_0(u)$ in $\left(W_p^1(\Omega)\right)'$, and $\lim_{j\to\infty}\langle A_0\,(u_j), u_j - u\rangle = 0$. Consequently, by (2.8) $(A(u_j)) \xrightarrow{w'} A(u)$ in $\left(W_p^1(\Omega)\right)'$ and by (2.7) we have

$$\lim_{j\to\infty} \langle A(u_j), u_j - u\rangle = 0.$$

So A is pseudomonotone operator which completes the proof of Lemma 1.

Lemma 2. *The operator*

$$A : W_p^1(\Omega) \to \left(W_p^1(\Omega)\right)' \quad \textit{is coercive, i.e.}$$

$$\lim_{\|u\|\to\infty} \frac{\langle A(u), u\rangle}{\|u\|} = +\infty.$$

Proof. From condition d) it follows that

$$\langle A_0(u), u\rangle \geq c_2'\|u\|^p_{W^1_p(\Omega)} - c_3', \tag{2.9}$$

where c_2', c_3' are positive constants. By assumption f) we have

$$\langle B_1(u), u\rangle \geq - \int_{\partial\Omega} h_1(x,u)u d\sigma \geq 0. \tag{2.10}$$

From assumption g) and Hölder's inequality we obtain

$$|\langle B_2(u), u\rangle \geq \tilde{c}_3\|u\|^{\rho_2+1}_{W^1_p(\Omega)} + c_4\|u\|_{W^1_p(\Omega)}, \tag{2.11}$$

where $\rho_2 + 1 < p$.

From condition i) we obtain

$$\begin{aligned}
|\langle B_3(u), u\rangle| &= \left| \int_{\partial\Omega} \left\{ \int_{\Gamma} \tilde{h}_3(x,\tau,u(\tau))d\sigma_\tau \right\} u(x)d\sigma_x \right| \leq \\
&\leq \int_{\partial\Omega} \left[\int_{\Gamma} \left|\tilde{h}_3(x,\tau,u(\tau))d\sigma_\tau\right| \right] |u(x)|d\sigma_x| \leq \\
&\leq \int_{\partial\Omega} \left[\int_{\Gamma} (c_3|u(\tau)|^{\rho_3} + k_3(\tau))d\sigma_\tau \right] |u(x)|d\sigma_x = \\
&= \left\{ \int_{\Gamma} (c_3|u(\tau)|^{\rho_3} + k_3(\tau))d\sigma_\tau \right\} \cdot \int_{\partial\Omega} |u(x)|d\sigma_x \leq \\
&\leq c_5 \left\{ \|u\|^{\rho_3}_{W^1_p(\Omega)} + \int_{\Gamma} k_3(\tau)d\sigma_\tau \right\} \|u\|_{W^1_p(\Omega)} \leq \\
&\leq c_5\|u\|^{\rho_3+1}_{W^1_p(\Omega)} + c_6\|u\|_{W^1_p(\Omega)},
\end{aligned}$$

where $\rho_3 + 1 < p$.

Consequently, by using (2.9), (2.11) we find

$$\frac{\langle A(u), u\rangle}{\|u\|} = \frac{\langle A_0(u), u\rangle}{\|u\|} - \frac{\langle B_1(u), u\rangle}{\|u\|} - \frac{\langle B_2(u), u\rangle}{\|u\|} - \frac{\langle B_3(u), u\rangle}{\|u\|} \geq$$

$$\geq c_2'\|u\|^p_{W_p^1(\Omega)} - c_3' - \tilde{c}_3\|u\|^{\rho_2+1}_{W_p^1(\Omega)} - c_4\|u\|_{W_p^1(\Omega)} - c_5\|u\|^{\rho_3+1}_{W_p^1(\Omega)} - c_6\|u\|_{W_p^1(\Omega)}.$$

From this inequality, by using $\rho_2 + 1 < p,\ \rho_3 + 1 < p$ it follows that

$$\lim_{\|u\|\to\infty} \frac{\langle A(u), u\rangle}{\|u\|} = +\infty.$$

The proof of Theorem 3. By the Lemmas 1 and 2 the operator $A : W_p^1(\Omega) \to (W_p^1(\Omega))'$ is pseudomonotone and coercive. By using the well-known theory of pseudomonotone operators in reflexive Banach spaces (see e.g. [9]) we obtain that for any $\tilde{F} \in (W_p^1(\Omega))'$ there exists $u \in W_p^1(\Omega)$ which satisfies (2.3) for all $v \in W_p^1(\Omega)$ with compact support and so the proof of existence theorem is complete.

Remark 2. It is possible to consider more general second order partial differential equations

$$\sum_{|\alpha|\leq 1} (-1)^{|\alpha|}\partial^\alpha f_\alpha(x, u, \partial_1 u, \dots, \partial_n u) + g(x, u) = F \quad \text{in} \quad \Omega$$

with nonlocal boundary condition (2.2), where in the terms $g(x,u)$ and $h_1(x,u)$ no growth restriction is imposed with respect to u but it is supposed that g, h_1 satisfy the sign conditions $g(x,\eta)\eta \geq 0$, $h_1(x,\eta)\eta \leq 0$ (see [7] and [8]).

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