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THE CHEBYSHEV COEFFICIENTS OF GENERAL-ORDER DERIVATIVES OF AN INFINITELY DIFFERENTIABLE FUNCTION IN TWO OR THREE VARIABLES

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Abstract. The tensor product of orthogonal Chebyshev polynomials is used to approximate a function of more than one variable. Expressions relating the Chebyshev coefficients of general-order derivatives of an infinitely differentiable function in two or three variables in terms of the original Chebyshev coefficients of the function are stated and proved.

1. Introduction

Spectral methods based on double Chebyshev polynomials for solving numerically partial differential equations have been used by many authors, among them, Dew and Scraton [2], Doha [3], Gottlieb and Orszag [5], Haidvogel and Zang [6] and Horner [7]. For solving high-order partial differential equations, for example the biharmonic equation, the Chebyshev coefficients of high derivatives of infinitely differentiable functions are required. In this paper general formulas for these coefficients are stated and proved. In Section 2 we give some properties of the double Chebyshev series expansions and in Section 3 we describe how they are used to solve Poisson's equation in two variables with the tau method as a model problem. In Section 4 we state and prove the main results of the paper which are three expressions for the coefficients of general order partial derivatives of an expansion in double Chebyshev polynomials in terms of the coefficients of the original expansion. Extension to expansion in triple Chebyshev polynomials is also considered in Section 5.

2. Properties of double Chebyshev series expansions

Let $u(x,y)$ be an infinitely differentiable function defined on the square S $(-1 \leq x, y \leq 1)$. Then it is possible to express

$$u(x,y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} {}'' a_{mn} T_m(x) T_n(y), \tag{1}$$

where $T_m(x), T_n(y)$ are Chebyshev polynomials of the first kind defined by

$$T_m(x) = \cos(m\cos^{-1}x), \; T_n(y) = \cos(n\cos^{-1}y).$$

Basu [1] refers to series (1) as a "bivariate" Chebyshev series expansion. The double primes in (1) indicate that the first term is $\frac{1}{4}a_{00}; a_{m0}$ and a_{0n} are to be taken as $\frac{1}{2}a_{m0}$ and $\frac{1}{2}a_{0n}$ for $m, n > 0$ respectively. Further, let

$$D_x^p D_y^q u(x,y) = u^{(p,q)}(x,y) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}{}'' a_{mn}^{(p,q)} T_m(x) T_n(y)$$

note here that we denote the Chebyshev expansion coefficients of the $p-th$ and $q-th$ partial derivatives of $u(x,y)$ with respect to x and y respectively by $a_{mn}^{(p,q)}$.

Following Orszag [9] and using the expressions (see, Fox and Parker [4])

$$2T_m(x) = \frac{1}{m+1} D_x T_{m+1}(x) - \frac{1}{m-1} D_x T_{m-1}(x), \quad m > 1, \tag{2}$$

$$2T_n(y) = \frac{1}{n+1} D_y T_{n+1}(y) - \frac{1}{n-1} D_y T_{n-1}(y), \quad n > 1 \tag{3}$$

with the assumptions that

$$D_x \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}{}'' a_{mn}^{(p-1,q)} T_m(x) T_n(y) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}{}'' a_{mn}^{(p,q)} T_m(x) T_n(y),$$

$$D_y \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}{}'' a_{mn}^{(p,q-1)} T_m(x) T_n(y) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}{}'' a_{mn}^{(p,q)} T_m(x) T_n(y)$$

it is possible to derive the expressions

$$a_{m-1,n}^{(p,q)} - a_{m+1,n}^{(p,q)} = 2m a_{mn}^{(p-1,q)}, \quad p \geq 1, \tag{4}$$

$$a_{m,n-1}^{(p,q)} - a_{m,n+1}^{(p,q)} = 2n a_{mn}^{(p,q-1)}, \quad q \geq 1. \tag{5}$$

Repeated application of (4) keeping n and q fixed (see, Phillips et al. [10]) yields

$$a_{mn}^{(p,q)} = 2\sum_{i=1}^{\infty}(m+2i-1) a_{m+2i-1,n}^{(p-1,q)}, \quad p \geq 1 \tag{6}$$

and the same with (5) keeping m and p fixed yields

$$a_{mn}^{(p,q)} = 2\sum_{j=1}^{\infty}(n+2j-1)a_{m,n+2j-1}^{(p,q-1)}, \quad q \geq 1. \tag{7}$$

3. The tau method for Poisson's equation in two dimensions

Consider Poisson's equation in the square S $(-1 \leq x, y \leq 1)$

$$D_x^2 u(x,y) + D_y^2 u(x,y) = f(x,y) \qquad -1 \leq x, y \leq 1 \tag{8}$$

with homogeneous Dirichlet boundary conditions

$$u(x,y) = 0 \qquad |x| = 1, |y| = 1 \tag{9}$$

and assume that both $u(x,y)$ and $f(x,y)$ are approximated by truncated double Chebyshev series

$$u(x,y) = \sum_{n=0}^{N}\sum_{m=0}^{M}{}'' a_{mn}T_m(x)T_n(y), \tag{10}$$

$$f(x,y) = \sum_{n=0}^{N}\sum_{m=0}^{M}{}'' f_{mn}T_m(x)T_n(y), \tag{11}$$

then the Chebyshev tau equations for Poisson's equation (8) are given by

$$a_{mn}^{(2,0)} + a_{mn}^{(0,2)} = f_{mn} \qquad 0 \leq m \leq M-2, \quad 0 \leq n \leq N-2 \tag{12}$$

while the Dirichlet boundary conditions (9) yield

$$\sum_{m=0}^{M}{}'(\pm 1)^m a_{mn} = 0, \qquad 0 \leq n \leq N, \tag{13}$$

$$\sum_{n=0}^{N}{}'(\pm 1)^n a_{mn} = 0, \qquad 0 \leq m \leq M. \tag{14}$$

The $2M+2N+4$ boundary conditions given by (13) and (14) are not all linearly independent; there exist four linear relations among them, namely

$$\sum_{n=0}^{N}\sum_{m=0}^{M}{}''(\pm 1)^m(\pm 1)^n a_{mn}=0. \tag{15}$$

Thus, equations (12), (13) and (14) give $(M+1)(N+1)$ equations for the $(M+1)(N+1)$ unknowns a_{mn} $(0\le m\le M, 0\le n\le N)$. The coefficients $a_{mn}^{(2,0)}$ and $a_{mn}^{(0,2)}$ of the second partial derivatives of the approximation $u(x,y)$ are related to the coefficients a_{mn} of $u(x,y)$ by invoking (6) with $p=1$ and $p=2$, and (7) with $q=1$, and $q=2$ respectively. In the next section we show how the coefficients of any derivatives may be expressed in terms of the original expansion coefficients. This allows us, for example, to replace $a_{mn}^{(2,0)}$ and $a_{mn}^{(0,2)}$ in (12) by an explicit expression in terms of the a_{mn}. In this way we can set up a linear system for a_{mn} $(0\le m\le M, 0\le n\le N)$ which may be solved using standard algorithms.

4. The theorem and its proof

Let $u(x,y)$ be an infinitely differentiable function defined on the square S $(-1\le x,y\le 1)$. The coefficients $a_{mn}^{(p,q)}$ of an expansion of double Chebyshev polynomials of the $p-th$ and $q-th$ partial derivatives of $u(x,y)$ with respect to x and y respectively are related to the coefficients $a_{mn}^{(0,q)}, a_{mn}^{(p,0)}$ and the original coefficients a_{mn} by:

$$a_{mn}^{p,q}= \tag{16}$$

$$=\frac{2^p}{(p-1)!}\sum_{i=1}^{\infty}\frac{(i+p-2)!(m+i+p-2)!}{(i-1)!(m+i-1)!}(m+2i+p-2)a_{m+2i+p-2,n}^{(0,q)},\qquad p\ge 1,$$

$$a_{mn}^{p,q}= \tag{17}$$

$$=\frac{2^q}{(p-1)!}\sum_{j=1}^{\infty}\frac{(j+q-2)!(n+j+q-2)!}{(j-1)!(n+j-1)!}(n+2j+q-2)a_{m,n+2j+q-2}^{(p,0)},\qquad q\ge 1,$$

$$a_{mn}^{(p,q)}= \tag{18}$$

$$=\frac{2^{p+q}}{(p-1)!(q-1)!}\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\frac{(i+p-2)!(m+i+p-2)!(j+q-2)!(n+j+q-2)!}{(i-1)!(m+i-1)!(j-1)!(n+j-1)!}\times$$

$$\times(m+2i+p-2)(n+2j+q-2)a_{m+2i+p-2,n+2j+q-2}, \quad p,q \geq 1,$$

for all $m, n \geq 0$.

In order to prove the theorem the following two lemmas are required:

(19)
$$\sum_{i=1}^{M}(m+2i-1)\frac{(M-i+p-1)!(m+i+M+p-2)!}{(M-i)!(m+i+M-1)!} = \\ = \frac{1}{p}\frac{(M+p-1)!(m+M+p-1)!}{(M-1)!(m+M-1)!}, \quad m \geq 0, p \geq 1,$$

(20)
$$\sum_{j=1}^{N}(n+2j-1)\frac{(N-j+q-1)!(n+j+N+q-2)!}{(N-j)!(n+j+N-1)!} = \\ = \frac{1}{q}\frac{(N+q-1)!(n+N+q-1)!}{(N-1)!(n+N-1)!}, \quad n \geq 0, q \geq 1.$$

The interested reader is referred to Karageorghis [8] for the proof of any of these two lemmas.

Proof of the theorem

Firstly, we prove formula (16). For $p = 1$, application of (6) with $p = 1$ yields the required formula. Proceeding by induction, assuming that the relation is valid for p (keeping n and q fixed), we want to show that,

(21)
$$a_{mn}^{(p+1,q)} = \\ = \frac{2^{p+1}}{p!}\sum_{i=1}^{M}\frac{(i+p-1)!(m+i+p-1)!}{(i-1)!(m+i-1)!}(m+2i+p-1)a_{m+2i+p-1,n}^{(0,q)}.$$

From (6) replacing p by $p+1$ and assuming the validity of (16) for p,

(22)
$$a_{mn}^{(p+1,q)} = \frac{2^{p+1}}{(p-1)!}\sum_{i=1}^{\infty}(m+2i-1)\left\{\sum_{k=1}^{\infty}\frac{(k+p-2)!(m+2i+k+p-3)!}{(k-1)!(m+2i+k-2)!}\times \right. \\ \left. \times (m+2i+2k+p-3)a_{m+2i+2k+p-3,n}^{(0,q)}\right\}.$$

Let $i+k = M+1$, then (22) takes the form

(23)
$$a_{mn}^{(p+1,q)} = \frac{2^{p+1}}{(p-1)!} \sum_{M=1}^{\infty} \left\{ \sum_{\substack{i,k=1 \\ i+k=M+1}}^{M} (m+2i-1) \frac{(k+p-2)!(m+2i+k+p-3)!}{(k-1)!(m+2i+k-2)!} \times \right.$$
$$\left. \times \, (m+2M+p-1) a_{m+2M+p-1,n}^{(0,q)} \right\}$$

which may also be written as

$$a_{mn}^{(p+1,q)} = \frac{2^{p+1}}{(p-1)!} \sum_{M=1}^{\infty} \left\{ \sum_{i=1}^{M} (m+2i-1) \frac{(M-i+p-1)!(m+i+M+p-2)!}{(M-i)!(m+i+M-1)!} \right\} \times$$
$$\times (m+2M+p-1) a_{m+2M+p-1,n}^{(0,q)}.$$

Application of lemma (19) to the second series yields equation (21) and the proof of formula (16) is complete. It can also be shown that formula (17) is true by following the same procedure with (7), keeping m and p fixed. Formula (18) is obtained immediately by substituting (16) into (17) or (17) into (16). This completes the proof of the theorem.

5. Extension to triple Chebyshev series expansions

Let $u(x,y,z)$ be an infinitely differentiable function defined on the cube $C\ (-1 \le x, y, z \le 1)$. Then it is possible to express

(24)
$$u(x,y,z) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{\ell=0}^{\infty}{}''' a_{\ell mn} T_\ell(x) T_m(y) T_n(z).$$

Further, let

(25)
$$D_x^p D_y^q D_z^r u(x,y,z) =$$
$$= u^{(p,q,r)}(x,y,z) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{\ell=0}^{\infty}{}''' a_{\ell mn}^{(p,q,r)} T_\ell(x) T_m(y) T_n(z).$$

Triple primes indicate that the first term is taken with factor 1/8; coefficients with two zero subscripts and with one zero subscript among the three subscripts ℓ, m and n are to be taken with factors $\frac{1}{4}$ and $\frac{1}{2}$ respectively. It can be easily shown that

(26)
$$a_{\ell-1,m,n}^{(p,q,r)} - a_{\ell+1,m,n}^{(p,q,r)} = 2\ell a_{\ell mn}^{(p-1,q,r)}, \qquad p \ge 1,$$

$$(27) \qquad a^{(p,q,r)}_{\ell,m-1,n} - a^{(p,q,r)}_{\ell,m+1,n} = 2m a^{(p,q-1,r)}_{\ell mn}, \qquad q \geq 1,$$

$$(28) \qquad a^{(p,q,r)}_{\ell,m,n-1} - a^{(p,q,r)}_{\ell,m,n+1} = 2n a^{(p,q,r-1)}_{\ell mn}, \qquad r \geq 1,$$

which, in turn, yield

$$(29) \qquad a^{(p,q,r)}_{\ell mn} = 2\sum_{i=1}^{\infty}(\ell + 2i - 1) a^{(p-1,q,r)}_{\ell+2i-1,m,n}, \qquad p \geq 1,$$

$$(30) \qquad a^{(p,q,r)}_{\ell mn} = 2\sum_{j=1}^{\infty}(m + 2j - 1) a^{(p,q-1,r)}_{\ell,m+2j-1,n}, \qquad q \geq 1,$$

$$(31) \qquad a^{(p,q,r)}_{\ell mn} = 2\sum_{k=1}^{\infty}(n + 2k - 1) a^{(p,q,r-1)}_{\ell,m,n+2k-1,}, \qquad r \geq 1.$$

Now, we state the following theorem, which is to be considered as an extension of the theorem given in Section 4.

Theorem. *The expansion coefficients* $a^{(p,q,r)}_{\ell mn}$ *of (25) are related to the coefficients with superscripts* $(0,q,r), (p,0,r), (p,q,0), (0,0,r), (0,q,0), (p,0,0)$ *and the original expansion coefficients* $a_{\ell mn}$ *of (24) by:*

$$(32) \qquad a^{(p,q,r)}_{\ell mn} =$$

$$= \frac{2^p}{(p-1)!}\sum_{i=1}^{\infty}\frac{(i+p-2)!(\ell+i+p-2)!}{(i-1)!(\ell+i-1)!}(\ell+2i+p-2) a^{(0,q,r)}_{\ell+2i+p-2,m,n}, \qquad p \geq 1,$$

$$(33) \qquad a^{(p,q,r)}_{\ell mn} =$$

$$= \frac{2^q}{(q-1)!}\sum_{j=1}^{\infty}\frac{(j+q-2)!(m+j+q-2)!}{(j-1)!(m+j-1)!}(m+2j+q-2) a^{(p,0,r)}_{\ell,m+2j+q-2,n}, \qquad q \geq 1,$$

$$(34) \qquad a^{(p,q,r)}_{\ell mn} =$$

$$= \frac{2^r}{(r-1)!}\sum_{k=1}^{\infty}\frac{(k+r-2)!(n+k+r-2)!}{(k-1)!(n+k-1)!}(n+2k+r-2) a^{(p,q,0)}_{\ell,m,n+2k+r-2}, \qquad r \geq 1,$$

(35)
$$a_{\ell mn}^{(p,q,r)} =$$
$$= \frac{2^{p+q}}{(p-1)!(q-1)!} \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \frac{(i+p-2)!(\ell+i+p-2)!(j+q-2)!(m+j+q-2)!}{(i-1)!(\ell+i-1)!(j-1)!(m+j-1)!} \times$$
$$\times (\ell+2i+p-2)(m+2j+q-2) a_{\ell+2i+p-2,m+2j+q-2,n}^{(0,0,r)}, \qquad p,q \geq 1,$$

(36)
$$a_{\ell mn}^{(p,q,r)} =$$
$$= \frac{2^{p+r}}{(p-1)!(r-1)!} \sum_{k=1}^{\infty} \sum_{i=1}^{\infty} \frac{(i+p-2)!(\ell+i+p-2)!(k+r-2)!(n+k+r-2)!}{(i-1)!(\ell+i-1)!(k-1)!(n+k-1)!} \times$$
$$\times (\ell+2i+p-2)(n+2k+r-2) a_{\ell+2i+p-2,m,n+2k+r-2}^{(0,q,0)}, \qquad p,r \geq 1,$$

(37)
$$a_{\ell mn}^{(p,q,r)} =$$
$$\frac{2^{q+r}}{(q-1)!(r-1)!} \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \frac{(j+q-2)!(m+j+q-2)!(k+r-2)!(n+k+r-2)!}{(j-1)!(m+j-1)!(k-1)!(n+k-1)!} \times$$
$$\times (m+2j+q-2)(n+2k+r-2) a_{\ell,m+2j+q-2,n+2k+r-2}^{(p,0,0)}, \qquad q,r \geq 1,$$

(38)
$$a_{\ell mn}^{(p,q,r)} =$$
$$= \frac{2^{p+q+r}}{(p-1)!(q-1)!(r-1)!} \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \frac{(i+p-2)!(\ell+i+p-2)!(j+q-2)!}{(i-1)!(\ell+i-1)!(j-1)!} \times$$
$$\times \frac{(m+j+q-2)!(k+r-2)!(n+2+r-2)!}{(m+j-1)!(k-1)!(n+k-1)!} \times$$
$$\times (\ell+2i+p-2)(m+2j+q-2)(n+2k+r-2) \times$$
$$\times a_{\ell+2i+p-2,m+2j+q-2,n+2k+r-2}, \qquad p,q,r \geq 1,$$

for all $\ell, m, n \geq 0$.

Outlines of the proof

The proof of formulas (32), (33) and (34) can be obtained by induction on p, q and r respectively. Substituting (32) into (33) and (34); and substituting (33) into

(34) yield immediately formulas (35), (36) and (37) respectively. Formula (38) is obtained by substituting (34) into (35), or (33) into (36), or (32) into (37).

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