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VIABILITY THEOREMS ON STRONGLY SLEEK TUBES

Z. Kánnai (Budapest, Hungary)

Abstract. In this paper we prove the existence of viable solutions to convex and nonconvex right-hand side differential inclusions with a time-dependent viability set. The time-dependence is related to some continuous differentiability.

1. Introduction

In this paper we give a solution to the viability problem

$$\begin{cases} x(t) & \in & P(t) \\ x'(t) & \in & F(t, x(t)) \\ x(0) & = & x_0 \end{cases} \tag{1}$$

when P is a time-dependent viability set. Viability theorems with a constant viability set are proven by Haddad [11] for autonomous differential inclusions and by Deimling [9], Ledyaev [16], Tallos [19] for nonautonomous inclusions with convex right-hand side; and with nonconvex right-hand side by Goncharov [10], Colombo [7] and Kánnai and Tallos [14]. We directly use the result of Colombo in [7] and the result of Tallos in [19]. We shortly describe these results. We denote by $\mathcal{F}(X)$ (resp. $\mathcal{F}_{conv}(X)$) the family of the nonempty, closed (and resp. convex) subsets of a Banach space X. $T_K(x)$ denotes the Bouligand contingent cone to a subset K at a point x. $C_K(x)$ means the Clarke cone (definitions see in [3]). We denote by Λ the σ-algebra of Lebesgue-measurable sets in $\mathbb{R}$, by $\mathcal{B}$ the Borel σ-algebra (in $\mathbb{R}$ or in a metric space, which is mostly a product of other metric spaces in this work), by B (resp. cl$B$) the open (resp. the closed) unit ball of a Banach space X.

Theorem 1 (Colombo). *Let X be a reflexive Banach space, denote the family of nonempty closed sets in X by $\mathcal{F}(X)$. Let us assume that*

(a) *K is a locally compact subset of X, $x_0 \in K$ is given;*

(b) *$F : [0,1] \times K \to \mathcal{F}(X)$ is a given set-valued map;*

(c) *F is $\Lambda \times \mathcal{B}$-measurable;*

(d) *$F(t, \cdot)$ is lower semicontinuous (l.s.c.) on K for a.e. $t \in [0,1]$;*

(e) *F is integrally bounded;*

(f) *$F(t,x) \subset T_K(x)$ for a.e. $t \in [0,1]$ and every $x \in K$.*

Then there exists a number $0 < T \leq 1$ and an absolutely continuous function $x : [0,T] \to X$ such that

$$
\left\{
\begin{array}{lll}
x(t) & \in & K \quad \text{for every} \quad t \in [0,1]; \\
x'(t) & \in & F(t, x(t)) \quad \text{for a.e.} \quad t; \\
x(0) & = & x_0.
\end{array}
\right. \tag{2}
$$

Theorem 2 (Tallos). *Let X be a finite dimensional Banach space, denote the family of nonempty closed convex sets in X by $\mathcal{F}_{conv}(X)$. Let us suppose that*

(a') *K is a closed subset of X, $x_0 \in K$ is given;*

(b') *$F : [0,1] \times K \to \mathcal{F}_{conv}(X)$ is a given set-valued map;*

(c') *$F(\cdot, x)$ is Λ-measurable for every $x \in K$;*

(d') *$F(t, \cdot)$ is upper semicontinuous (u.s.c.) on K for a.e. t;*

(e') *F is integrally bounded;*

(f') *$F(t,x) \cap C_K(x) \neq \emptyset$ for a.e. $t \in [0,1]$ and every $x \in K$.*

Then for every $T > 0$ there exists a viable solution to (2).

To be able to use these theorems, we have to make some remarks about measurability. In order to do it, introduce a new definition.

Definition 1. Let K be a separable metric space. Define the set $\mathcal{S} := \{A \subset \mathbb{R} \times K : \exists\, B \in \mathcal{B}(\mathbb{R}), \lambda(B) = 0$ such that $A \backslash (B \times K)$ is Borel$\}$.
It can be easily seen that $\mathcal{S}$ is a σ-field on $\mathbb{R} \times K$. It can be also seen that $\Lambda \times \mathcal{B} \subset \mathcal{S}$ because whenever $A \in \Lambda$ and $B \in \mathcal{B}(K)$, there exists a set $A_0 \in \mathcal{B}(\mathbb{R})$ such that $A_0 \subset A$ and $\lambda(A \backslash A_0) = 0$. Of course, the set $A_0 \times B$ is Borel. Consequently, the set $A \times B \in \mathcal{S}$.

Proposition 1. *Let X be a separable Banach space and K be a separable metric space, $F : [0,1] \times K \to \mathcal{F}(X)$ be $\mathcal{S}$-measurable and integrally bounded with a function $\ell \in L^1[0,1]$. Then there exists a map $F_1 : [0,1] \times K \to \mathcal{F}(X)$ $\Lambda \times \mathcal{B}$-measurable and integrally bounded and a Borel-measurable set $A \subset [0,1]$, $\lambda(A) = 0$, such that for every $t \in [0,1] \backslash A$ and $x \in K$ it holds*

$$F_1(t,x) = F(t,x).$$

Proof. Let $(G_i)_{i \in \mathbb{N}}$ be a countable basis of the space X. Denote

$$H_i := \{(\tau, y) \in [0,1] \times K : F(\tau, y) \cap G_i \neq \emptyset\}.$$

Since F is $\mathcal{S}$-measurable, the sets $H_i \in \mathcal{S}$. Then for every i there exists a set $A_i \in \mathcal{B}(\mathbb{R})$ such that $\lambda(A_i) = 0$ and $H_i \backslash (A_i \times X)$ is Borel. Denote $A := \bigcup_{i=1}^{\infty} A_i$. Then A is Borel and $\lambda(A) = 0$. Denote

$$F_1(t,x) := \begin{cases} F(t,x) & \text{if } t \notin A; \\ \ell(t) \cdot \mathrm{cl} B & \text{if } t \in A. \end{cases}$$

Let an open set G be given in X. Then there exists an index set $\mathcal{P} \subset \mathbb{N}$ such that $G = \cup\{G_i : i \in \mathcal{P}\}$. Then

$$\begin{aligned} &\{(t,x) \in [0,1] \times K : F_1(t,x) \cap G \neq \emptyset\} = \\ &= \{(t,x) \in ([0,1]\backslash A) \times K : F(t,x) \cap G \neq \emptyset\} \cup \\ &\cup \{(t,x) \in A \times K : G \cap \ell(t) \cdot \mathrm{cl} B \neq \emptyset\}. \end{aligned}$$

The second member of the union is obviously $\Lambda \times \mathcal{B}$-measurable. Examine the first one:

$$\begin{aligned} &\{(t,x) \in ([0,1]\backslash A) \times K : F(t,x) \cap G \neq \emptyset\} = \\ &= \{(t,x) : F(t,x) \cap G \neq \emptyset\} \backslash A \times K = \\ &= \bigcup_{i \in \mathcal{P}} \{(t,x) : F(t,x) \cap G_i \neq \emptyset\} \backslash A \times K = \\ &= \bigcup_{i \in \mathcal{P}} H_i \backslash A \times K = \bigcup_{i \in \mathcal{P}} (H_i \backslash A \times K), \end{aligned}$$

and the sets $H_i \backslash (A \times K)$ are obviously Borel. So the desired set is also Borel. By this we have shown that the map F_1 is $\Lambda \times \mathcal{B}$-measurable.

2. Premilinaries

Consider a Banach space X. A set $K \in \mathcal{F}(X)$ we call *sleek* if its Bouligand cone as a map

$$T_K : K \to \mathcal{F}(X), \quad x \mapsto T_K(x)$$

is l.s.c. We note that $T_K(x) = C_K(x)$ whenever K is sleek (see Lemma 10 in Ch.7 in [4]) where $C_K(x)$ denotes the Clarke tangent cone. It follows that whenever K is sleek, the cone $T_K(x)$ will be convex.

Let a map $P : [0,1] \to \mathcal{F}(X)$ be given. We will say that P is *sleek* if graph P is sleek in the space $\mathbb{R} \times X$.

A set K is called ***boundedly compact*** if $n \cdot \mathrm{cl} B \cap K$ is compact for every n. We note that if the set K in Theorem 1 is boundedly compact, then K is separable, moreover the number T can be chosen equal to 1.

We also need the idea of contingent derivative.

Definition 2. For every $(t,x) \in \text{graph } P$ and $u \in \mathbb{R}$ denote

$$DP(t,x)(u) := \{v \in X : (u,v) \in T_{\text{graph}P}(t,x)\}.$$

The map DP we will call the contingent derivative of the map P.

Definition 3. A map $P : [0,1] \to \mathcal{F}(X)$ is called *strongly sleek* if it is sleek, moreover there exists a constant $M > 0$ such that

$$DP(t,x)(1) \cap M \cdot \text{cl}B \neq \emptyset$$

for every $t \in [0,1)$ and $x \in P(t)$.
We get an interesting characterization of strongly sleek tubes by the following proposition.

Proposition 2. *A sleek map $P : [0,1] \to \mathcal{F}(X)$ with a boundedly compact graph is strongly sleek if and only if there exists a constant $M > 0$ such that for every $(t,x) \in \text{graph } P$ there exists a continuously differentiable selection φ from P such that $\varphi(t) = x$ and $\|\varphi'(t)\| \leq M$ for every $t \in [0,1]$.*

Proof. The necessity is obvious, because

$$\varphi'(t) \in DP(t,x)(1),$$

so

$$DP(t,x)(1) \cap M \cdot \text{cl}B \neq \emptyset.$$

Now consider a strongly sleek map $P : [0,1] \to \mathcal{F}(X)$ with a boundedly compact graph. Then the map $T_{\text{graph}P} = C_{\text{graph}P}$ is convex valued and l.s.c. Since $DP(t,x)(1) \cap (2M+1)\text{cl}B$ is nonempty, it will be also l.s.c. (see Proposition 4 later) and convex valued. So by Michael's selection theorem (see in [17]) we get that it has a continuous selection $\hbar$: graph $P \to X$. Then by definition

$$F : \text{graph } P \to \mathcal{K}(\mathbb{R} \times X), \quad F\big((t,x)\big) := \big\{\big(1, \hbar(t,x)\big)\big\}$$

from [14] we obtain that there exists an absolutely continuous function $y = (\tau, x) : [t_0, 1] \to$ graph P such that $y'(t) \in F\big(y(t)\big)$ and $y(t_0) = (t_0, x_0)$ for any $(t_0, x_0) \in$ graph P. It yields $\tau(t) = t$, so $x'(t) = \hbar\big(t, x(t)\big)$. Furthermore with an obvious modification in a similar way we get that there exists an absolutely continuous function $z : [0, t_0] \to$ graph P such that $z'(t) = \hbar\big(t, z(t)\big)$ and $z(t_0) = x_0$. Then the linked function φ is absolutely continuous, $\varphi(t_0) = x_0$ and $\varphi'(t) = \hbar\big(t, \varphi(t)\big)$. Since

$(t, \varphi(t)) \in$ graph P, the function φ is a selection from P. Moreover $\hbar$ is continuous, thus, φ' is continuous. Finally,

$$\|\varphi'(t)\| = \|\hbar(t, \varphi(t))\| \leq 2M + 1.$$

Of course, it means that a strongly sleek map is Lipschitz-continuous with the mentioned constant $2M + 1$ as a Lipschitz-constant. In finite dimension the Lipschitzeanity by itself is enough to follow the strongly sleekness of a sleek map.

Proposition 3. *Let X be a finite dimensional Banach space. A sleek map $P : [0,1] \to \mathcal{F}(X)$ with a closed graph is strongly sleek if and only if P is Lipschitz-continuous, i.e. there exists a constant $M > 0$ such that for every $t_1, t_2 \in [0,1]$*

$$P(t_1) \subset P(t_2) + M \cdot |t_2 - t_1| \cdot \mathrm{cl}B$$

and

$$P(t_2) \subset P(t_1) + M \cdot |t_2 - t_1| \cdot \mathrm{cl}B$$

holds true.

Proof. The necessity we have seen. Take a Lipschitz-continuous sleek map P, let $\{h_n\}$ be a sequence of numbers converging to 0 from above and let $t \in [0,1)$ be a number such that

$$t + h_n \in [0,1] \qquad (n \in \mathbb{N}).$$

Let $x \in P(t)$ be a given vector. Then there exists a vector $y_n \in P(t + h_n)$ such that

$$\|y_n - x\| \leq d(x, P(t + h_n)) \leq M \cdot h_n.$$

Denote $v_n := \dfrac{y_n - x}{h_n}$, then $\|v_n\| \leq M$. So by the Bolzano-Weierstrass theorem we get that there exists a subsequence $(v_{nk}) \subset (v_n)$ and a vector $v \in X$ such that $v_{nk} \to v$. Thus $\|v\| \leq M$. On the other hand,

$$d\left(v, \frac{P(t + h_{nk}) - x}{h_{nk}}\right) \leq d\left(v, \frac{v_{nk} - x}{h_{nk}}\right) = d(v, v_{nk}) \to 0.$$

Thus, using Proposition 2 in Ch.4. sect.3 in [3], we have

$$v \in DP(t,x)(1).$$

So

$$DP(t,x)(1) \cap M \cdot \mathrm{cl}B \neq \emptyset$$

for every $t \in [0,1)$ and $x \in P(t)$.

Proposition 4. *Let X be a Banach space and $P : [0,1] \to \mathcal{F}(X)$ be a strongly sleek map with a constant $M > 0$. Then for every constant $c \geq 1$ the map*

$$H_c : \text{graph } P \to \mathcal{F}(X)$$
$$H_c(t,x) := DP(t,x)(1) \cap (2M + c) \cdot \text{cl}B$$

is l.s.c. Moreover the map $DP(\cdot,\cdot)(1)$ is also l.s.c.

Proof. Consider a couple $(t,x) \in \text{graph } P$. Let $G \subset X$ be a given open set such that

$$H_c(t,x) \cap G \neq \emptyset.$$

Then there exists a vector $v \in H_c(t,x) \cap G$ and a positive number ε such that

$$v + 20(M + c)\varepsilon \cdot \text{cl}B \subset G.$$

Since $T_{\text{graph}P}$ is l.s.c., there exists a neighbourhood U of (t,x) such that whenever $(\tau, y) \in U$ there will exist a couple

$$(r,u) \in T_{\text{graph}P}(\tau,y) \cap \big((1,v) + \varepsilon \cdot (-1,1) \times B\big)$$

(we note that the set $(-1,1) \times B$ is the unit ball in $\mathbb{R} \times X$ considered with the maximum norm). Thus, $|r - 1| < \varepsilon$ and $\|u - v\| < \varepsilon$. So $|r| < 1 + \varepsilon$ and $\|u\| < 2M + c + \varepsilon$ since $v \in H_c(t,x)$. Obviously we get $\frac{u}{r} \in DP(\tau,y)(1)$ and

$$\left\|\frac{u}{r} - v\right\| \leq \|u - v\| + \left\|\frac{u}{r} - u\right\| < \varepsilon + \left|\frac{1}{r} - 1\right| \cdot \|u\| \leq$$
$$\leq \varepsilon + \frac{|r-1|}{r}(2M + c + \varepsilon) \leq 2\varepsilon(2M + c + 1),$$

provided we suppose that $\varepsilon < \frac{1}{2}$ because in that case $r > \frac{1}{2}$. Then $\left\|\frac{u}{r} - v\right\| < (4M + 2c + 2)\varepsilon$ so $\left\|\frac{u}{r}\right\| \leq 2M + c + (4M + 2c + 2)\varepsilon$. On the other hand, there exists a vector $u_0 \in DP(\tau,y)(1)$ such that $\|u_0\| \leq M$. Then $(1,u_0)$ and $\left(1,\frac{u}{r}\right)$ belong to $T_{\text{graph}P}(\tau,y) = C_{\text{graph}P}(\tau,y)$ since P is sleek. Thus, from convexity of $C_{\text{graph}P}(\tau,y)$ we get

$$\left(1, (1 - 4\varepsilon)\frac{u}{r} + 4\varepsilon u_0\right) \in C_{\text{graph}P}(\tau,y),$$

so we have

$$(*) \qquad \hat{u} := (1 - 4\varepsilon)\frac{u}{r} + 4\varepsilon u_0 \in DP(\tau,y)(1)$$

provided $\varepsilon \le \frac{1}{4}$. Now,

$$\|\hat{u}\| \le (1-4\varepsilon)\left\|\frac{u}{r}\right\| + 4\varepsilon\|u_0\| \le$$

$$\le (1-4\varepsilon)\big(2M + c + (4M + 2c + 2)\varepsilon\big) + 4\varepsilon M \le$$

$$\le 2M + c + (4M + 2c + 2)\varepsilon - 8\varepsilon M - 4\varepsilon c - (16M + 8c + 8)\varepsilon^2 + 4\varepsilon M \le$$

$$\le 2M + c + 2(1-c)\varepsilon \le 2M + c,$$

since $c \ge 1$. Thus, $\|\hat{u}\| \le 2M + c$. Then by $(*)$ we get

$$\hat{u} \in H_c(\tau, y).$$

On the other hand,

$$\|v - \hat{u}\| \le \left\|v - \frac{u}{r}\right\| + \left\|\frac{u}{r} - \hat{u}\right\| <$$

$$< (4M + 2c + 2)\varepsilon + 4\varepsilon\left(\left\|\frac{u}{r}\right\| + \|u_0\|\right) \le$$

$$4(M + c)\varepsilon + 4\varepsilon\big(3M + c + 4(M + c)\varepsilon\big) \le 20(M + c)\varepsilon,$$

ε having been smaller than $\frac{1}{4}$.

Thus, since $v + 20(M + c)\varepsilon B \subset G$, we get $\hat{u} \in G$, i.e.

$$(**) \qquad H_c(\tau, y) \cap G \ne \emptyset.$$

Now we have proven that $(**)$ holds true for every $(\tau, y) \in U$. It just means the lower semicontinuity of H_c.

Now examine the map $DP(\cdot,\cdot)(1)$. If $G \subset X$ is an open subset such that $DP(t,x)(1) \cap G \ne \emptyset$, then there exists a number $c > 1$, such that $H_c(t,x) \cap G \ne \emptyset$. We have seen that H_c is l.s.c. So in a neighbourhood of (t,x) we get $H_c(\tau,y) \cap G \ne \emptyset$. Of course, in this case $DP(\tau,y)(1) \cap G \ne \emptyset$.

3. The nonconvex case

Consider a reflexive Banach space X and a strongly sleek map $P : [0,1] \to \mathcal{F}(X)$ with a boundedly compact graph. Let F : graph $P \to \mathcal{F}(X)$ be a given map such that

(α) F is $\Lambda \times \mathcal{B}$-measurable;

(β) F is l.s.c. on $P(t)$ for a.e. $t \in [0,1]$;

(γ) F is integrally bounded, i.e. there exists a function $\ell \in L^1[0,1]$, $\ell \geq 0$ such that for a.e. $t \in [0,1]$ and every $x \in P(t)$

$$F(t,x) \subset \ell(t) \cdot B;$$

(δ) $F(t,x) \subset DP(t,x)(1)$ for a.e. $t \in [0,1]$ and every $x \in P(t)$.

Theorem 3. *Let us assume that the conditions written above are satisfied. Then for any $x_0 \in P(0)$ there exists a function $x \in AC([0,1];X)$ such that*

$$\begin{cases} x(t) & \in & P(t) \text{ for every } t \in [0,1]; \\ x'(t) & \in & F\big(t, x(t)\big) \text{ for a.e. } t, \\ x(0) & = & x_0. \end{cases} \tag{3}$$

Proof. Denote $\hat{P} := \text{graph } P \subset \mathbb{R} \times X$. We can obviously assume that $1 \leq \ell(t)$ for a.e. t. Define the map $\hat{F} : [0,1] \times \hat{P} \to \mathcal{F}(\mathbb{R} \times X)$ by

$$\hat{F}\big(t,(\tau,x)\big) := \begin{cases} \{(1,v) : v \in F(t,x)\}, & \text{if } t = \tau; \\ \{(1,v) : v \in H_{\ell(t)}(\tau,x)\}, & \text{if } t \neq \tau; \end{cases}$$

where $H_{\ell(t)}(\tau,x) := DP(\tau,x)(1) \cap \big(2M + \ell(t)\big) \cdot \text{cl}B$ and M is a constant of strongly sleekness of P. Then $\hat{\ell}(t) := 2M + \ell(t)$ is an integrable function such that

$$\hat{F}\big(t,(\tau,x)\big) \subset \hat{\ell}(t) \cdot \hat{B}$$

for a.e. t and every $(\tau,x) \in \hat{P}$ where $\hat{B}$ is the unit ball of the Banach space $\mathbb{R} \times X$ since $\ell(t) \geq 1$. On the other hand, by Proposition 4, we get that the map $H_{\ell(t)}$ is l.s.c. for a.e. $t \in [0,1]$. Thus, from (δ) we have that $\hat{F}\big(t,(\cdot,\cdot)\big)$ is l.s.c. for a.e. t, since the set

$$\big\{\big(t,(t,x)\big) : x \in P(t)\big\} \subset \{t\} \times \hat{P}$$

is closed. Furthermore from the inclusion

$$H_{\ell(t)}(\tau,x) \subset DP(\tau,x)(1)$$

and from (δ) we get

$$\hat{F}\big(t,(\tau,x)\big) \subset T_{\hat{P}}(\tau,x)$$

for a.e. $t \in [0,1]$ and every $(\tau,x) \in \hat{P}$. Now we show that $\hat{F}$ is $\mathcal{S}$-measurable where $\mathcal{S}$ means the σ-field mentioned in the first paragraph with the metric space $\hat{P}$ and the Banach space $\mathbb{R} \times X$.

Denote $Q := \{(t,(t,x)) : (t,x) \in \hat{P}\}$. It is a closed set in $[0,1] \times \hat{P}$. Now take an open subset G in $\mathbb{R} \times X$. Then the set

$$Z := \{(t,x) \in \hat{P} : (1, F(t,x)) \cap G \neq \emptyset\}$$

is obviously $\Lambda \times \mathcal{B}(X)$-measurable since F is measurable. So there exists a Borel-set $A \subset [0,1]$, $\lambda(A) = 0$ such that the set $Z \backslash (A \times X)$ is Borel-measurable in the space $\mathbb{R} \times X$. Then

$$\{(t,(\tau,x)) \in [0,1] \times \hat{P} : \hat{F}(t,(\tau,x)) \cap G \neq \emptyset\} =$$

$$= \Big(\big\{(t,(\tau,x)) : (1, H_{\ell(t)}(\tau,x)) \cap G \neq \emptyset\big\} \backslash Q\Big) \bigcup \{(t,(t,x)) : (1, F(t,x)) \cap G \neq \emptyset\}.$$

This set will be obviously $\mathcal{S}$-measurable if we show that the set

$$W := \{(t,(t,x)) : (1, F(t,x)) \cap G \neq \emptyset\}$$

is $\mathcal{S}$-measurable. It is carried out in the following way:

$$\begin{aligned} \{(t,(t,x)) : (1, F(t,x)) \cap G \neq \emptyset\} &= \\ = \{(t,(t,x)) : (t,x) \in Z\} &= \\ = \{(t,(t,x)) : (t,x) \in Z \backslash (A \times X)\} &\cup E \end{aligned}$$

where the set $E \subset A \times (\mathbb{R} \times X)$. Since the function

$$\begin{aligned} f : Q &\to \hat{P} \\ (t,(t,x)) &\mapsto (t,x) \end{aligned}$$

is continuous, the set $\{(t,(t,x)) : (t,x) \in Z \backslash (A \times X)\}$ is Borel-measurable in $\mathbb{R} \times (\mathbb{R} \times X)$. Thus, from definition of $\mathcal{S}$ we immediately get that W is $\mathcal{S}$-measurable. So we have shown the $\mathcal{S}$-measurability of the set

$$\{(t,(\tau,x)) \in [0,1] \times \hat{P} : \hat{F}(t,(\tau,x)) \cap G \neq \emptyset\},$$

which means the $\mathcal{S}$-measurability of $\hat{F}$. According to (δ), without loss of generality we can assume that X is separable, because $(1, F(t,x)) \subset \operatorname{cl} \operatorname{lin} \operatorname{graph} P$. Then the space $\mathbb{R} \times X$ is also separable. Now by Proposition 1 there exists a $\Lambda \times \mathcal{B}$-measurable and integrally bounded map $\hat{F}_1 : [0,1] \times \hat{P} \to \mathcal{F}(\mathbb{R} \times X)$ such that for a.e. t and every (τ, x)

$$\hat{F}_1(t,(\tau,x)) = \hat{F}(t,(\tau,x)).$$

Now the conditions of Theorem 1 for the map $\hat{F}_1$ with the modifications mentioned in the second paragraph are fulfilled. Thus, there exists a function $y \in AC([0,1];\mathbb{R} \times X)$ such that

$$\begin{cases} y(t) & \in & \hat{P} \text{ for every } t \in [0,1]; \\ y'(t) & \in & \hat{F}_1(t, y(t)) \text{ a.e. } t \in [0,1]; \\ y(0) & = & (0, x_0). \end{cases} \tag{4}$$

Of course, in this case we can write $\hat{F}$ instead of $\hat{F}_1$. That is, there exists a function $\tau \in AC([0,1];\mathbb{R})$ and a function $x \in AC([0,1];X)$ such that $y = (\tau, x)$, so

$$\begin{cases} \big(\tau(t), x(t)\big) \in & \text{graph } P; \\ \big(\tau(t), x(t)\big) \in & \hat{F}\Big(t, \big(\tau(t), x(t)\big)\Big); \\ \tau(0) = 0, & x(0) = x_0. \end{cases} \tag{5}$$

Then from definition $\hat{F}$ we get that $\tau'(t) = 1$ for a.e. $t \in [0,1]$ and $\tau(0) = 0$. It follows that $\tau(t) = t$ for every $t \in [0,1]$ since τ is absolutely continuous. Then from (5) we have that

$$\big(t, x(t)\big) \in \text{ graph } P \text{ for every } t \in [0,1]$$

and

$$\big(1, x(t)\big) \in \hat{F}\Big(t, \big(t, x(t)\big)\Big) \text{ for a.e. } t \in [0,1],$$

which just means that x is a solution to (3).

By Proposition 3 we immediately get the next theorem from the previous one:

Theorem 4. *Let X be a finite dimensional Banach space and $P : [0,1] \to \mathcal{F}(X)$ be a sleek and Lipschitz-continuous map with a closed graph. Let $F :$ graph $P \to \mathcal{F}(X)$ be a set-valued map satisfying (α) – (δ). Then there exists a solution $x \in AC([0,1];X)$ to (3).*

4. The convex case

In this section we give a viability theorem with a weaker tangential condition than in the previous one, however we have to suppose, as usual, that the values of F are convex. Unfortunately, we must assume that $F(t, \cdot)$ is Hausdorff-continuous for a.e. t instead of the usual upper semicontinuity because of the method of our proof. Let us assume that X is a finite dimensional Euclidean space and $P : [0,1] \to \mathcal{F}_{conv}(X)$ is a strongly sleek map with a closed graph.

Moreover suppose that $F :$ graph $P \to \mathcal{F}_{conv}(X)$ is a given map such that

(α') F is $\Lambda \times \mathcal{B}$-measurable on graph P;

(β') $F(t,\cdot)$ is continuous on $P(t)$ for a.e. $t \in [0,1]$;

(γ') F is integrally bounded;

(δ') $F(t,x) \cap DP(t,x)(1) \neq \emptyset$ for a.e. $t \in [0,1]$ and every $x \in P(t)$.

Theorem 5. *Let us assume that the conditions (α') - (δ') are satisfied. Then for every $x_0 \in P(0)$ there exists a solution $x \in AC([0,1];X)$ to (3).*

Proof. We use the notations of Theorem 3. Since $P(t)$ is convex, for any $x \in X$ there exists one and only one point $x_0 \in P(t)$ such that $\|x - x_0\|$ is minimal. Denote this x_0 by $\Pi_{P(t)}(x)$. Then $\Pi_{P(t)} : X \to P(t)$ is a continuous function. Define

$$\Gamma(t,x) := F\big(t, \Pi_{P(t)}(x)\big) \quad (t \in [0,1],\ x \in X).$$

Then Γ is measurable in t and continuous in x.

Define for $t \in [0,1]$ and $(\tau, x) \in \hat{P}$

$$\varepsilon\big(t,(\tau,x)\big) := \inf\big\{d\big(v,\Gamma(t,x)\big) : v \in DP(\tau,x)(1)\big\}.$$

Since $DP(\cdot,\cdot)(1)$ is l.s.c., we have $\varepsilon\big(t,(\tau,x)\big) = \inf\big\{d\big(f(\tau,x),\Gamma(t,x)\big) : f$ is a continuous selection from $DP(\cdot,\cdot)(1)\big\}$ by Michael's selection theorem (see in [17]). Obviously for each f the function $(\tau,x) \mapsto d\big(f(\tau,x),\Gamma(t,x)\big)$ is continuous. Thus, the infimum, i.e. $\varepsilon\big(t,(\cdot,\cdot)\big)$, is upper semicontinuous. It can be easily seen that ε is measurable in t (see Lemma III.39 in [6]). On the other hand, in case $t = \tau$ we have $\varepsilon(t,t,x) = 0$ because of (δ'). Since $\|F(t,x)\| \leq \ell(t)$ and $DP(\tau,x)(1) \cap M \cdot \mathrm{cl}B \neq \emptyset$, we get

$$\varepsilon\big(t,(\tau,x)\big) \leq \ell(t) + M.$$

Define for $t \in [0,1]$ and $(\tau,x) \in \hat{P}$

$$F_*\big(t,(\tau,x)\big) := \Gamma(t,x) + \varepsilon(t,\tau,x) \cdot \mathrm{cl}B.$$

Then F_* has convex closed values and it is u.s.c. in (τ,x) and measurable in t. F_* is obviously integrally bounded with the function $2\ell(t) + M$. Due to the definition we get

$$F_*\big(t,(\tau,x)\big) \cap DP(\tau,x)(1) \neq \emptyset. \tag{6}$$

If $t = \tau$, then $x \in P(t)$. So by definition Γ and ε we have

$$F_*\big(t,(t,x)\big) = \Gamma(t,x) = F(t,x).$$

Denote

$$\hat{F}\big(t,(\tau,x)\big) := \big\{(1,v) : v \in F_*\big(t,(\tau,x)\big)\big\}.$$

Then the map $\hat{F} : [0,1] \times \hat{P} \longrightarrow \mathcal{F}_{conv}(\mathbb{R} \times X)$ is measurable in t and u.s.c. in (τ, x) and integrally bounded. By (6) we get

$$\hat{F}(t, (\tau, x)) \cap T_{\hat{P}}(\tau, x) \neq \emptyset,$$

on the other hand, because of the sleekness of $\hat{P}$ we obtain

$$\hat{F}(t, (\tau, x)) \cap C_{\hat{P}}(\tau, x) \neq \emptyset.$$

Thus, $\hat{F}$ satisfies all the conditions of Theorem 2. Consequently, there exists a function $y = (\tau, x) \in AC([0,1]; \mathbb{R} \times X)$ which is a solution to (5). Now the rest of the proof is carried out in the same way as in Theorem 3.

Similarly to Theorem 4, we obtain the following theorem.

Theorem 6. *Let X be a finite dimensional Banach space and $P : [0,1] \longrightarrow \mathcal{F}_{conv}(X)$ be a Lipschitz-continuous sleek map with a closed graph. Let us assume that $F :$ graph $P \longrightarrow \mathcal{F}_{conv}(X)$ is a set-valued map satisfying (α') - (δ'). Then for each $x_0 \in P(0)$ there exists a viable solution to (3).*

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Z. Kánnai
Eötvös Loránd University, Computer Center
H-1117 Budapest, XI. Bogdánfy u. 10/b.
Hungary