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SOME NOTES ON MULTISTEP ITERATION METHODS

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Abstract. This paper generalizes well-known results given previously for single-step iterations. Three types of convergence conditions are presented. Contraction-type results are first given for multistep iterations, for both cases of explicit and implicit processes. Then, conditions for the monotone convergence of the explicit method are introduced, and in the third part of the paper generalized Newton-type methods are analysed.

1. Introduction

During the last years an increasing attention has been given to the convergence analysis of iteration procedures for solving nonlinear equations (see for example Schmidt [4], Potra and Ptak [2], Potra and Rheinboldt [3]). Most of the works are concerned with single-step methods, but multistep iteration processes are very important not only

for computing solutions of nonlinear equations, but also in investigating stability of equilibria in dynamic games. An application of multistep iterations in such problems is presented by Szidarovszky and Okuguchi (1987).

In this paper the well-known convergence and monotone convergence criteria (Ortega and Rheinboldt [1]) will be generalized for multistep processes. The development of this paper is as follows. In Section 2 the convergence of explicit and implicit multistep processes is investigated. In Section 3 conditions for the monotone convergence of the same processes are discussed, and in Section 4 generalized monotone Newton-type iterarions are analysed.

2. Convergence criteria

In this section the convergence of the iteration process

$$x_{k+1} = G_{k+1}(x_k, \dots, x_{k+1-\ell}) \tag{2.1}$$

will be analysed, where $x_0, x_1, \dots, x_{\ell-1}$ are the initial approximations.

Let $D \subset R^n, G_{k+1} : D^\ell \to R^n, G : D^\ell \to R^n$ (where $D^\ell = D \times D \times \ldots \times D$). Assume that there exists a set D_0 such that $\bar{D}_0 \subset D$, furthermore

(a) $x_k \in D_0$ for all $k \geq 0$;

(b) for all $s_1, s_2, \ldots, s_{\ell+1} \in D_0$, and $k \geq 0$,

$$\| G_{k+1}(s_1, s_2, \ldots, s_\ell) - G_k(s_2, \ldots, s_\ell, s_{\ell+1}) \| \leq \sum_{i=1}^{\ell} \alpha_i \| s_i - s_{i+1} \|, \tag{2.2}$$

(c) $\sum_{i=1}^{n} \alpha_i < l, \quad \alpha_i \geq 0 \quad (i = l, 2, \ldots, \ell)$;

(d) for all $x \in D, \quad G_k(x, \ldots, x) \to G(x, \ldots, x) \quad (k \to \infty)$.

The main convergence result can be given as

Theorem 1. *Under conditions (a)-(d) the iteration process (2.1) converges to the unique solution of equation* $x=G(x,\ldots,x)$ *in* $\bar{D}_0$.

Since the proof of this theorem is a simple extension of the proof of Weinitschke [6], the details are omitted.

Remark. The stationary version of the theorem has been proven by Weinitschke [6]. The single step version of this theorem is given in Ortega and Rheinboldt ([1] p. 389).

In many applications the new iterate x_{k+1} cannot be expressed as an explicit function of the previous iterates $x_k, \ldots, x_{k+1-\ell}$. In such cases a generalized version of the iteration scheme (2.1) is applied. This generalized implicit iteration is as follows:

$$x_{k+1} = G_{k+1}(x_{k+1}, x_k, \ldots, x_{k+1-\ell}), \tag{2.3}$$

where $G_{k+1} : D^{\ell+1} \to R^n$. Assume that there exists a subset $D_o \subset D$ such that

(i) For arbitrary $s_1, \ldots, s_{\ell+1}, t_1, \ldots, t_{\ell+1} \in D_0$,

$$| G_k(s_{\ell+1}, \ldots, s_1) - G_k(t_{\ell+1}, \ldots, t_1) | \leq \sum_{i=1}^{\ell+1} P_i \mid s_i - t_i \mid,$$

where for any vector $u = (u_i), | u | = (| u_i |)$, and the matrices P_i are nonnegative;

(ii) Let $T = \sum_{i=1}^{\ell+1} P_i$ and $P = \begin{pmatrix} Q_\ell & Q_{\ell-1} & & Q_2 & Q_1 \\ I & & & & \\ & I & & & \\ & & & & \\ & & & I & 0 \end{pmatrix}$

with $Q_i \doteq (I - P_{\ell+1})^{-1} P_i$, then $\rho(T)$ and $\rho(P)$ are less than unity. Here $\rho(\cdot)$ denotes the spectral radius of real matrices;

(iii) Define $H_k : D \to R^n, H_k(x) = | G_k(x, \ldots, x) - G(x) |$, where $G(x) = G(x, \ldots, x)$ and assume that for all $x \in D_0$ and $k \to \infty, H_k(x) \to 0$. (Here $G : D^\ell \to R^n$);

(iv) Assume that initial approximations $x_0, \ldots, x_{\ell-1} \in D_0$ are selected so that equation $x = G_\ell(x, x_{\ell-1}, \ldots, x_0)$ has a unique solution x_ℓ in D_0, furthermore

$$S = \{x \in R^n \mid | x - x_\ell | \leq \varepsilon\} \subset D_0,$$

where ε is selected so that

$$T\varepsilon + 2v + \sum_{i=1}^{\ell} P_i | x_\ell - x_{i-1} | \leq \varepsilon \tag{2.4}$$

with $v \geq H_k(x_\ell), k \geq \ell$.

The convergence of the implicit scheme (2.3) is guaranteed by

Theorem 2. *If assumptions (i), (ii), (iii) and (iv) hold, then for all* $k \geq \ell$ *implicit iteration* $x = G_k(x, x_{k-1}, \ldots, x_{k-\ell})$ *has a unique solution* $x_k \in S, \lim x_k = x^*$, *where* $x^* \in S$ *is the unique fixed point of* G *in* D_0, *that is* $x^* = G(x^*, x^*, \ldots, x^*)$.

Proof. We shall proceed in several stages.

(a) Let $x, y \in D_0$ arbitrary, then for all $k \geq 1$,

$$| G(x) - G(y) | \leq | G(x) - G_k(x, \ldots, x) | + | G_k(x, \ldots, x) - G_k(y, \ldots, y) | +$$

$$+ | G_k(y, \ldots, y) - G(y) | \leq H_k(x) + H_k(y) + \sum_{i=1}^{\ell+1} P_i | x - y | .$$

For $k \to \infty$ the first two terms tend to zero, and therefore $\rho(\sum_{i=1}^{\ell+1} P_i) < 1$ implies that $G(\cdot)$ is a P-contraction on D_0 (see Ortega and Rheinboldt [1]).

(b) We shall now show that $G(\cdot)$ maps S into itself. Let $x \in S$, then

$$| G(x) - x_\ell | \leq | G(x) - G(x_\ell) | + | G(x_\ell) - G_\ell(x_\ell, \ldots, x_\ell) | +$$

$$+ | G_\ell(x_\ell, \ldots, x_\ell) - G_\ell(x_\ell, x_{\ell-1}, \ldots, x_0) | \leq T \cdot | x - x_\ell | + H_\ell(x_\ell) +$$

$$+ \sum_{i=1}^{\ell} P_i | x_\ell - x_{i-1} | \leq T \cdot \varepsilon + v + \sum_{i=1}^{\ell} P_i | x_\ell - x_{i-1} | \leq \varepsilon .$$

Hence $x = G(x)$ has a unique fixed point $x^*(\in S)$ on D_0.

(c) Next we verify that for $x, y_1, \ldots, y_\ell \in S$, $G_k(x, y_\ell, \ldots, y_1) \in S$ $(k \geq \ell)$. For this purpose consider the following inequality:

$$\begin{aligned}
&| G_k(x, y_\ell, \ldots, y_1) - x_\ell | \leq \\
&\leq | G_k(x, y_\ell, \ldots, y_1) - G_k(x_\ell, \ldots, x_\ell) | + | G_k(x_\ell, \ldots, x_\ell) - G(x_\ell) | + \\
&+ | G(x_\ell) - G_\ell(x_\ell, \ldots, x_\ell) | + | G_\ell(x_\ell, \ldots, x_\ell) - G_\ell(x_\ell, x_{\ell-1}, \ldots, x_0) | \leq \\
&\leq \sum_{i=1}^{\ell} P_i | y_i - x_\ell | + P_{\ell+1} | x - x_\ell | + H_k(x_\ell) + H_\ell(x_\ell) + \sum_{i=1}^{\ell} P_i | x_\ell - x_{i-1} | \leq \\
&\leq T\varepsilon + 2v + \sum_{i=1}^{\ell} P_i | x_\ell - x_{i-1} | \leq \varepsilon.
\end{aligned}$$

Hence equation $x = G_k(x, y_\ell, \ldots, y_1)$ is a P-contraction on S for any fixed $y_\ell, \ldots, y_1 \in S$, therefore it has a unique solution in S, which is the unique solution in D_0.

(d) Finally, consider

$$\begin{aligned}
&| x_k - x^* | \leq | G_k(x_k, x_{k-1}, \ldots, x_{k-\ell}) - G_k(x^*, \ldots, x^*) | + \\
&+ | G_k(x^*, \ldots, x^*) - G(x^*) | \leq \sum_{i=1}^{\ell+1} P_i | x_{k-\ell+i-1} - x^* | + H_k(x^*).
\end{aligned}$$

Denote $u_k = | x_k - x^* |$, then we have

$$u_k \leq \sum_{i=1}^{\ell-1} P_i u_{k-\ell+i-1} + v_k,$$

where $v_k = H_k(x^*)$. Since $P_{\ell+1} \leq T, \rho(P_{\ell+1}) < 1$ and $(I - P_{\ell+1})^{-1}$ is nonnegative. Hence

$$u_k \leq (I - P_{\ell+1})^{-1} \sum_{i=1}^{\ell} P_i u_{k-\ell+i-1} + w_k$$

with $w_k = (I - P_{\ell+1})^{-1} v_k$. Define $z_{k-1}^{(1)} = u_{k-1}, \ldots, z_{k-1}^{(\ell)} = u_{k-\ell}$,

$$z_{k-1} = \begin{pmatrix} z_{k-1}^{(1)} \\ \vdots \\ z_{k-1}^{(\ell)} \end{pmatrix}, \qquad s_k = \begin{pmatrix} w_k \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Then

$$z_k \le P z_{k-1} + s_k.$$

Since $\rho(P) < 1$, there exists a norm such that $\| P \| < 1$. Let $\varepsilon > 0$ arbitrary, and assume that $\| s_k \| \le \varepsilon$ for $k \ge K$, and define $k_0 = \max\{K, \ell\}$. Then

$$z_k \le P^i z_{k-i} + s_k + P s_{k-1} + \ldots + P^{i-1} s_{k+1-i} \qquad (i \ge 1).$$

If one selects $k_0 = k + 1 - i$, that is, $i = k + 1 - k_0$, then we have

$$z_k \le P^{k+1-k_0} z_{k_0-1} + s_k + P s_{k-1} + \ldots + P^{k-k_0} s_{k_0}.$$

Hence

$$\| z_k \| \le \| P \|^{k+1-k_0} \| z_{k_0-1} \| + (\| I \| + \| P \| + \ldots + \| P \|^{k-k_0}) \cdot \varepsilon \le$$

$$\le \| P \|^{k+1-k_0} \| z_{k_0-1} \| + \frac{1}{1 - \| P \|} \cdot \varepsilon,$$

which shows that $\| z_k \| \to 0$, so $z_k \to 0$ and therefore $u_k \to 0$, that is $x_k \to x^*$. Thus, the proof is completed.

Remarks.

1. Condition (iii) implies that sequence $\{H_k(x_\ell)\}$ is bounded, therefore there exists a vector v which satisfies (iv). Inequality (2.4) is equivalent to

$$(I - T)\varepsilon \ge 2v + \sum_{i=1}^{\ell} P_i \mid x_\ell - x_{i-1} \mid.$$

We can easily show that (2.4) is satisfied by selecting

$$\varepsilon_0 = (I - T)^{-1} \Big[2v + \sum_{i=1}^{\ell} P_i \mid x_\ell - x_{i-1} \mid \Big]. \tag{2.5}$$

Since $\rho(T) < 1$ and $T \ge 0$, matrix $(I - T)^{-1}$ is nonnegative. Consequently $\varepsilon_0 \ge 0$. Furthermore premultiplying (2.5) by $(I - T)$, we conclude that (2.4) is satisfied with equality.

2. Consider now the special case of $\ell = 1$, which corresponds to one-step iterations. Condition (ii) now means that

$$\rho(P_2) < 1, \ \rho(P_2 + P_1) < 1, \ \text{ and } \ \rho(P) < 1,$$

where $P = Q_1 = (I - P_2)^{-1}P_1$. We can easily show that $\rho(P) < 1$ and $\rho(P_2) < 1 \Rightarrow \rho(P_2 + P_1) < 1$. For, observe that $P_2 + P_1 \geq 0$, furthermore

$$I - P_1 - P_2 = (I - P_2)(I - (I - P_2)^{-1}P_1) = (I - P_2)(I - P).$$

This implies

$$(I - P_1 - P_2)^{-1} = (I - P)^{-1}(I - P_2)^{-1} \geq 0,$$

and therefore $\rho(P_1 + P_2) < 1$.

This case of $\ell = 1$ corresponds to the single-step iteration process, which was analysed in Ortega and Rheinboldt [1]. Our theorem for the multistep process is a straightforward generalization of that classical result.

3. If G_{k+1} and G do not depend on their first argument, then theorem gives conditions for the convergence for explicit iterations.

3. Monotone convergence of multistep iterations

In this section sufficient conditions will be given for the monotone convergence of the iteration scheme (2.1). In R^n we say that $x \leq y$ if and only if $x_i \leq y_i$, where x_i and y_i are the components of x and y, respectively.

Definition 1. Mapping $G : D^\ell \to R^n$ is called *increasingly isotone* on D, if for arbitrary $s_i \in D$ $(i = 1, 2, \ldots, \ell + 1)$ such that $s_{\ell+1} \geq s_\ell \geq \ldots \geq s_2 \geq s_1$,

$$G(s_{\ell+1}, s_\ell, \ldots, s_2) \geq G(s_\ell, s_{\ell-1}, \ldots, s_1). \tag{3.1}$$

In the literature we say that mapping G is *isotone*, if $x_i, x_i' \in D, x_i \leq x_i'$ $(i = 1, 2, \ldots, \ell)$ imply $G(x_\ell, \ldots, x_1) \leq G(x_\ell', \ldots, x_1')$. In analysing monotone convergence of iteration processes it is usually assumed that mapping G is isotone. We can however easily show that if G is isotone, then it is also increasingly isotone, but if G is increasingly isotone, then it is not necessarily isotone. The first statement is obvious, and the second statement is verified by the following example.

Example 1. Define

$$D = [0, 1] \subset R^1,$$

and let

$$G(x, y) = \begin{cases} x, & \text{if } y \geq 2x - 1, \\ y - x + 1, & \text{if } y < 2x - 1. \end{cases}$$

Note that the function is continuous on D^2.

First we show that G is increasingly isotone. Select $x \geq y \geq z$, then we have to prove that $G(x, y) \geq G(y, z)$. First observe that $G(y, z) \leq y$. For, if $z \geq 2y - 1$, then

$G(y,z) = y$, and if $z < 2y-1$, then $G(y,z) = z-y+1 < 2y-1-y+1 = y$. On the other hand, we can prove that $G(x,y) \geq y$. For if $y \geq 2x-1$, then $G(x,y) = x \geq y$, and if $y < 2x-1$, then $G(x,y) = y-x+1 \geq y$, since $x \leq 1$.

But G is not isotone. Take points $(1,y)$ and $(1-\varepsilon, y)$ where $0 < y < 1-2\varepsilon < 1$. Then $G(1,y) = y$, and $G(1-\varepsilon, y) = y-1+\varepsilon+1 = y+\varepsilon$. That is, by decreasing the first variable the functional value increases.

This example shows that the property "increasingly isotone" is more general than the property "isotone" of functions.

(A) Assume that mapping G is increasingly isotone on D.

Consider the iteration scheme

$$x_{k+1} = G(x_k, \dots, x_{k+1-\ell}) \tag{3.2}$$

starting from initial approximations $x_i \in D (0 \leq i \leq \ell - 1)$, such that $x_0 \leq x_1 \leq \dots \leq x_\ell$. Assume that $x_k \in D$ $(k \geq \ell)$.

Theorem 3. *In the case of the iteration (3.2), $x_{k+1} \geq x_k$ for all $k \geq 0$.*

Proof. By induction, assume that for i $(i < k), x_{i+1} \geq x_i$. Then (3.1) implies that

$$x_{k+1} = G(x_k, \dots, x_{k+1-\ell}) \geq G(x_{k-1}, \dots, x_{k-\ell}) = x_k.$$

Since the assertion is true for $k < \ell + 1$, the theorem is proven.

Consider next the iteration scheme

$$y_{k+1} = G(y_{k+1-\ell}, \dots, y_k) \tag{3.3}$$

starting from initial approximations $y_i \in D$ $(0 \leq i \leq \ell - 1)$, such that $y_0 \geq y_1 \geq \dots \geq y_\ell$. Assume that $y_k \in D$ $(k \geq \ell)$.

Theorem 4. *In the case of the iteration (3.3), $y_{k+1} \leq y_k$ for all $k \geq 0$.*

Proof. Assume again that for i $(i < k), y_{i+1} \leq y_i$. Then, from (3.1) we may conclude that

$$y_{k+1} = G(y_{k+1-\ell}, \dots, y_k) \leq G(y_{k-\ell}, \dots, y_{k-1}) = y_k.$$

Since the assertion holds for $k < \ell + 1$, the theorem is proven.

Before presenting further conditions for the monotonicity of iteration sequences a lemma is presented, it can be proven by the repeated application of (3.1).

Lemma 1.*Assume that G is increasingly isotone on D, then for arbitrary $s_\ell \geq s_{\ell-1} \geq \dots \geq s_1 \geq t_\ell \geq t_{\ell-1} \geq \dots \geq t_1$ $(t_i \in D,\ s_i \in D,\ i = 1, \dots, \ell)$,*

$$G(s_\ell, s_{\ell-1}, \dots, s_1) \geq G(t_\ell, t_{\ell-1}, \dots, t_1).$$

(B) Assume now that mappings $K : D^\ell \to R^n$ and $H : D^\ell \to R^n$ are increasingly isotone on D. Starting from initial approximations

$$x_0 \le x_1 \le \ldots \le x_{\ell-1} \le y_{\ell-1} \le \cdots \le y_1 \le y_0$$

($x_0 \in D, y_0 \in D$) consider the iteration sequences

$$\begin{aligned} x_{k+1} &= K(x_k, x_{k-1}, \ldots, x_{k+1-\ell}) - H(y_{k+1-\ell}, \ldots, y_{k-1}, y_k) \\ y_{k+1} &= K(y_{k+1-\ell}, \ldots, y_{k-1}, y_k) - H(x_k, x_{k-1}, \ldots, x_{k+1-\ell}). \end{aligned} \tag{3.4}$$

Assume that $x_{\ell-1} \le x_\ell \le y_\ell \le y_{\ell-1}$ and

$$< x_0, y_0 >= \{x \mid x \in R^n, \quad x_0 \le x \le y_0\} \subset D.$$

Theorem 5. *Under the above assumptions*

a) $x_k \le x_{k+1} \le y_{k+1} \le y_k \qquad (k \ge 0)$;

b) if z is a fixed point of $G = K - H$ in $< x_\ell, y_\ell >$, then for all k, $x_k \le z \le y_k$.

Proof. By induction, assume that $x_{i+1} \ge x_i$ and $y_{i+1} \le y_i$ $(i < k)$. Then obviously

$$K(x_k, x_{k-1}, \ldots, x_{k+1-\ell}) \ge K(x_{k-1}, x_{k-2}, \ldots, x_{k-\ell})$$

and

$$H(y_{k+1-\ell}, y_{k+2-\ell}, \ldots, y_k) \le H(y_{k-\ell}, y_{k-\ell-1}, \ldots, y_{k-1}).$$

By subtracting these inequalities, we obtain that $x_{k+1} \ge x_k$. Similarly by subtracting the first equation from the second one after exchanging functions K and H we conclude that $y_{k+1} \le y_k$.

Next we shall prove that for all k, $x_k \le y_k$. Assume now using finite induction again, that $x_i \le y_i$ $(i \le k)$. Then

$$\begin{aligned} x_{k+1} &= K(x_k, x_{k-1}, \ldots, x_{k+1-\ell}) - H(y_{k+1-\ell}, y_{k+2-\ell}, \ldots, y_k) \le \\ &\le K(y_{k+1-\ell}, y_{k+2-\ell}, \ldots, y_k) - H(x_k, x_{k-1}, \ldots, x_{k+1-\ell}) = y_{k+1}, \end{aligned}$$

which proves the assertion for $k+1$.

Assume now that z is a fixed point of G in $< x_\ell, y_\ell >$, then by induction we can verify that for all $k > \ell$, $x_k \le z \le y_k$. For, assume that $z \in < x_i, y_i >$ $(i \le k)$, then

$$\begin{aligned} x_{k+1} &= K(x_k, x_{k-1}, \ldots, x_{k+1-\ell}) - H(y_{k+1-\ell}, y_{k+2-\ell}, \ldots, y_k) \le \\ &\le K(z, z, \ldots, z) - H(z, z, \ldots, z) = G(z, z, \ldots, z) \le \end{aligned}$$

$$\leq K(y_{k+1-\ell}, y_{k+2-\ell}, \ldots, y_k) - H(x_k, x_{k-1}, \ldots, x_{k+1-\ell}) = y_{k+1},$$

which completes the proof of the theorem.

Corollary (Generalized Kantorovich Lemma). *Assume that G is increasingly isotone on D, and the initial approximations satisfy $x_0 \leq x_1 \leq \ldots \leq x_\ell \leq y_\ell \leq \ldots \leq y_2 \leq y_1$, and assume that $x_0, y_0 \in D$, $< x_0, y_0 > \subset D$. Then for (3.2) and (3.3), $x_k \uparrow x^*$ and $y_k \downarrow y^*$ for some $x^*, y^* \in D, x^* \leq y^*$. Moreover, if G is continuous (as an ℓ-variable function) on $< x_0, y_0 > \times \ldots \times < x_0, y_0 >$ $(=< x_0, y_0 >^\ell)$, then x^* and y^* are fixed points of G and any fixed point $z \in < x_\ell, y_\ell >$ satisfies $z \in < x^*, y^* >$.*

Proof. Selecting $K = G$ and $H = 0$ Theorem 5 implies that for all $k \geq 1, x_k \leq y_0$ and $y_k \geq x_0$. Then sequences $\{x_k\}$ and $\{y_k\}$ are convergent, $x_k \uparrow x^*$ and $y_k \downarrow y^*$. Since for all k, $x_k \leq y_k$, we have that $x^* \leq y^*$. For continuous G the iterations

$$x_{k+1} = G(x_k, x_{k-1}, \ldots, x_{k+1-\ell})$$

$$y_{k+1} = G(y_{k+1-\ell}, y_{k+2-\ell}, \ldots, y_k)$$

imply that x^* and y^* are fixed points. Statement b) of Theorem 5 implies that for all k,

$$x_k \leq z \leq y_k,$$

where $z \in < x_\ell, y_\ell >$ is any fixed point of G. Thus

$$x^* \leq z \leq y^*,$$

which completes the proof.

Remark. The above theorems remain valid in more general function spaces and partial orders.

4. Monotone Newton-type iterations

In this section monotone Newton-type iterarions will be investigated.

(A) Consider equation $F(y, \ldots, y) = 0$, where $F : D^\ell \to R^n$. Assume that there exists a mapping $A : D^\ell \to R^n$ such that

$$F(s_{\ell-1}, \ldots, s_0) - F(s_\ell, \ldots, s_1) \leq A(s_{\ell-1}, \ldots, s_0)(s_{\ell-1} - s_\ell) \tag{4.1}$$

for all $s_i \in D$, $s_i \leq s_{i-1} \leq \ldots \leq s_1 \leq s_0$.

Consider now the iteration scheme

$$y_{k+1} = y_k - P_k(y_k, \ldots, y_{k+1-\ell})F(y_k, \ldots, y_{k+1-\ell}) \tag{4.2}$$

where $P_k(\cdot)$ is a nonnegative right subinverse of $A(\cdot)$. Assume that the initial approximations from D are selected so that $y_{\ell-1} \le y_{\ell-2} \le \ldots \le y_1 \le y_0$ and $F(y_{\ell-1},\ldots,y_0) \ge 0$. Then we are able to verify

Theorem 6. *If $y_k \in D$ for all $k \ge 0$, then*

a) $y_k \le y_{k-1}$ for all k, $F(y_k,\ldots,y_{k+1-\ell}) \ge 0$ for all $k \ge \ell - 1$.

b) If sequence $\{y_k\}$ is bounded below, that is $y_k \ge x$ for some $x \in R^n$, then there exits an y^* such that $y_k \downarrow y^*$.

c) If in addition, there exists a nonsingular $P \in L(R^n)$ such that

$$P_k(y_k,\ldots,y_{k+1-\ell}) \ge P \ge 0, \quad (\forall k \ge k_0)$$

and F is continuous at $(y^*,\ldots,y^*)$, then $F(y^*,\ldots,y^*) = 0$.

Proof. (a) First we shall prove by induction that a) holds. The statement holds for $k \le \ell - 1$. Assume that it is valid for a given k. Then (4.2) implies that $y_{k+1} \le y_k$, furthermore

$$\begin{aligned} &F(y_{k+1},\ldots,y_{k+2-\ell}) \ge \\ &\ge F(y_k,\ldots,y_{k+1-\ell}) + A(y_k,\ldots,y_{k+1-\ell})(y_{k+1} - y_k) = \\ &= [I - A(y_k,\ldots,y_{k+1-\ell})P_k(y_k,\ldots,y_{k+1-\ell})]F(y_k,\ldots,y_{k+1-\ell}) \ge 0. \end{aligned}$$

Thus the induction completed.

(b) If sequence $\{y_k\}$ is bounded from below, then the monotonicity of the sequence implies the convergence.

(c) Iteration (4.2) implies that for $k \ge k_0$,

$$y_k - y_{k+1} = P_k(y_k,\ldots,y_{k+1-\ell})F(y_k,\ldots,y_{k+1-\ell}) \ge P \cdot F(y_k,\ldots,y_{k+1-\ell}) \ge 0.$$

Since $y_k - y_{k+1} \to 0$ for $k \to \infty$, and P is invertable, $F(y_k,\ldots,y_{k+1-\ell}) \to 0$. The continuity of F at $(y^*,\ldots,y^*)$ implies that $F(y_k,\ldots,y_{k+1-\ell}) \to F(y^*,\ldots,y^*) = 0$.

(B) A monotonically increasing sequence will be now constructed. Assume that there exists a mapping $B : D^\ell \to R^n$ such that

$$F(s_{\ell-1},\ldots,s_0) - F(s_\ell,\ldots,s_1) \le B(s_{\ell-1},\ldots,s_0)(s_0 - s_1) \tag{4.3}$$

for all $s_i \in D$, $s_\ell \le s_{\ell-1} \le \ldots \le s_1 \le s_0$.

Consider now the iteration scheme

$$x_{k+1} = x_k - Q_k(x_{k+1-\ell},\ldots,x_k)F(x_{k+1-\ell},\ldots,x_k), \tag{4.4}$$

where $Q_k(\cdot)$ is a nonnegative right subinverse of $B(\cdot)$. Assume that the initial approximations from D are selected so that $x_{\ell-1} \geq x_{\ell-2} \geq \ldots \geq x_1 \geq x_0$ and $F(x_0, \ldots, x_{\ell-1}) \leq 0$. Then we can prove the following

Theorem 7. *If $x_k \in D$ for all $k \geq 0$, then*

(a) $x_k \geq x_{k-1}$ for all k, $F(x_{k+1-\ell}, \ldots, x_k) \leq 0$ (for $k \geq \ell - 1$);

(b) If sequence $\{x_k\}$ is bounded from above, that is, $x_k \leq y$ for some $y \in R^n$ then there exists an x^* such that $x_k \uparrow x^*$;

(c) If in additon, there exists a nonsigular $Q \in L(R^n)$ such that

$$Q_k(x_{k+1-\ell}, \ldots, x_k) \geq Q \geq 0 \qquad (\forall k \geq k_0)$$

and F is continuous at $(x^*, \ldots, x^*)$, then $F(x^*, \ldots, x^*) = 0$.

Proof. Assertion (a) can be again proven by induction. If $k = \ell - 1$, then the assertion is true. Assume now that it holds for a k. Then (4.4) implies that $x_{k+1} \geq x_k$. Furthermore from (4.3) we have that

$$F(x_{k+2-\ell}, \ldots, x_{k+1}) \leq F(x_{k+1-\ell}, \ldots, x_k) + B(x_{k+1-\ell}, x_{k+2-\ell}, \ldots, x_k) \cdot (x_{k+1} - x_k)$$
$$= (I - B(x_{k+1-\ell}, \ldots, x_k)Q_k(x_{k+1-\ell}, \ldots, x_k))F(x_{k+1-\ell}, \ldots, x_k) \leq 0,$$

thus the induction is completed.

(b) If $\{x_k\}$ is bounded from above, then $x_k \uparrow x^*$ for some x^*.

(c) Since for $k \geq k_0$,

$$x_{k+1} - x_k = Q_k(x_{k+1-\ell}, \ldots, x_k)(-F(x_{k+1-\ell}, \ldots, x_k)) \geq$$
$$\geq -QF(x_{k+1-\ell}, \ldots, x_k) \geq 0,$$

and $x_{k+1} - x_k \rightarrow 0$ for $k \rightarrow \infty$, we may conclude that for $k \rightarrow \infty$ $F(x_{k+1-\ell}, \ldots, x_k) \rightarrow F(x^*, \ldots, x^*) = 0$.

Remark. If $x^* = y^*$, then Theorems 6 and 7 imply that for all $k \geq 0$, $x_k \leq x^* \leq y_k$. Consequently,

$$\max\{| x_k - x^* |, | y_k - x^* |\} \leq | y_k - x_k |,$$

which gives a practically applicable error estimate for the iteration error of x_k and y_k.

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