Annales Univ. Sci. Budapest., Sect. Comp. 13 (1992) 25-33

TRIADDITIVE FUNCTIONS

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1. Introduction

Let Λ be the set of the strictly decreasing sequences $\lambda = (\lambda_n)$ of positive real numbers for which $L(\lambda) := \sum_{n=1}^{\infty} \lambda_n < +\infty$. A sequence $(\lambda_n) \in \Lambda$ is called *interval filling* if for any $x \in [0, L(\lambda)]$ there exists a sequence (δ_n) such that $\delta_n \in \{0,1\}$ for all $n \in \mathrm{N}$ and $x = \sum_{n=1}^{\infty} \delta_n \lambda_n$.This concept was introduced and discussed in [1]. It is known from [1] that $\lambda = (\lambda_n) \in \Lambda$ is interval filling if and only if $\lambda_n \leq L_{n+1}(\lambda)$ for all $n \in \mathrm{N}$ where $L_m(\lambda) = \sum_{i=m}^{\infty} \lambda_i, m \in \mathrm{N}$. The set of the interval filling sequences will be denoted by IF.

An *algorithm* (with respect to $\lambda = (\lambda_n) \in IF$) is defined as a sequence of functions $\alpha_n : [0, L(\lambda)] \to \{0,1\}$ ($n \in \mathrm{N}$) for which

$$x = \sum_{n=1}^{\infty} \alpha_n(x)\lambda_n$$

for all $x \in [0, L(\lambda)]$. We denote the set of algorithms (with respect to $\lambda = (\lambda_n) \in IF$) by $\mathcal{A}(\lambda)$. Obviously, $\mathcal{A}(\lambda) \neq \emptyset$ for all $\lambda \in IF$. Namely, it was proved in [1], [2] and [3] that if $\lambda = (\lambda_n) \in IF$ and

$$(1.1) \qquad \varepsilon_n(x) = \begin{cases} 0 & \text{if} \quad x < \sum_{i=1}^{n-1} \varepsilon_i(x)\lambda_i + \lambda_n \\ 1 & \text{if} \quad x \geq \sum_{i=1}^{n-1} \varepsilon_i(x)\lambda_i + \lambda_n, \qquad n \in \mathrm{N}, \quad x \in [0, L(\lambda)] \end{cases}$$

or

$$(1.2) \qquad \varepsilon_n^*(x) = \begin{cases} 0 & \text{if} \quad x \leq \sum_{i=1}^{n-1} \varepsilon_i^*(x)\lambda_i + \lambda_n \\ 1 & \text{if} \quad x > \sum_{i=1}^{n-1} \varepsilon_i^*(x)\lambda_i + \lambda_n, \qquad n \in \mathrm{N}, \quad x \in [0, L(\lambda)] \end{cases}$$

or

$$\text{(1.3)}\qquad \varepsilon_n^{\circ}(x) = \begin{cases} 0 & \text{if } \ x \le \sum_{i=1}^{n-1} \varepsilon_i^{\circ}(x)\lambda_i + L_{n+1}(\lambda) \\ 1 & \text{if } \ x > \sum_{i=1}^{n-1} \varepsilon_i^{\circ}(x)\lambda_i + L_{n+1}(\lambda), \end{cases} \qquad n \in \mathrm{N}, \quad x \in [0, L(\lambda)]$$

then $\varepsilon = (\varepsilon_n), \varepsilon^* = (\varepsilon_n^*), \varepsilon^{\circ} = (\varepsilon_n^{\circ}) \in \mathcal{A}(\lambda)$. $\varepsilon, \varepsilon^*$ and ε° are called the *regular, quasiregular* and *antiregular algorithm*, respectively.

If (a_n) is a sequence in $\mathbf{R}$ such that $\sum_{n=1}^{\infty} \mid a_n \mid < +\infty$, $\lambda = (\lambda_n) \in IF$, $\mathcal{A}_0 \subset \mathcal{A}(\lambda)$, $\mathcal{A}_0 \neq \emptyset$, $F : [0, L(\lambda)] \to \mathbf{R}$ and

$$F(x) = \sum_{n=1}^{\infty} \alpha_n(x) a_n, \qquad x \in [0, L(\lambda)]$$

for all $(\alpha_n) \in \mathcal{A}_0$ then F will be called an $\mathcal{A}_0$-*additive* function (with respect to λ). It is known that the $\mathcal{A}(\lambda)$-additive functions are linear [4] and an $\{\varepsilon\}$-additive function is continuous if and only if it is $\{\varepsilon, \varepsilon^*\}$-additive (is called *biadditive*)[2]. But there is such a biadditive function which is nowhere differentiable in $[0, L(\lambda)]$ (see in [5]). In [6], in case of smooth interval filling sequences, Z. Daróczy and I. Kátai proved that if the biadditive function is positive for positive values of the variable, or differentiable on a set of positive measure, then it is linear.

In this paper we will prove, for special interval filling sequences, that if the biadditive function is $\{\varepsilon^{\circ}\}$-additive then it is linear. The $\{\varepsilon, \varepsilon^*, \varepsilon^{\circ}\}$-additive function is called *triadditive*.

2. Triadditive functions

Theorem. *Let $\lambda = (\lambda_n) \in IF$ such that*

$$\text{(2.1)}\qquad \lambda_n \ge \lambda_{n+1} + \lambda_{n+2}$$

holds for every $n \in \mathrm{N}$. Moreover let $F : [0, L(\lambda)] \to \mathrm{R}$ be a triadditive function with respect to this λ, then F is linear.

Proof. For every $x \in [0, L(\lambda)]$ let

$$F^*(x) := F(x) - \frac{F(L(\lambda))}{L(\lambda)} x.$$

Then F^* is also a triadditive function with respect to λ, and using the notions $a_n := F^*(\lambda_n)$ $(n \in \mathrm{N})$

$$F^*(0) = F^*(L(\lambda)) = \sum_{n=1}^{\infty} a_n = 0.$$

We will prove that $F^* \equiv 0$. Suppose that the function $F^* \not\equiv 0$ and let

$$P_+ := \{n \in \mathrm{N} \mid a_n > 0\}, \quad P_- := \{m \in \mathrm{N} \mid a_m < 0\},$$

then $P_+ \cup P_- \neq \emptyset$, and thus, by $F^*(L(\lambda)) = 0$ the P_+, P_- are not empty sets. Moreover these are infinite sets. For example if P_+ is a finite set then let $n := \max P_+$. Now

$$\lambda_n = \sum_{i=n+1}^{\infty} \varepsilon_i^*(\lambda_n)\lambda_i,$$

whence by the triadditivity of F^* we have

$$a_n = \sum_{i=m+1}^{\infty} \varepsilon_i^*(\lambda_n)a_i \leq 0,$$

and this contradicts the inequality $a_n > 0$. That is, P_+ is an infinite set. The infinite property of P_- could be proved in similar.

For our proof we shall need the following two lemmas:

Lemma 1. *If $n \in P_+$ and $\lambda_n \neq L_{n+1}$ then there exists $n_1 > n$, $n_1 \in P_+$ such that*

$$a_n \leq \sum_{i=n+1}^{n_1} a_i. \tag{2.2}$$

Proof of the Lemma 1. Let $n \in P_+$ such that

$$\lambda_n = \sum_{i=n+1}^{n+s(n)} \lambda_i,$$

where, by (2.1) $s(n) \in \mathrm{N}$, $s(n) \geq 2$. In this case, by (1.2) the quasiregular expansion of λ_n is

$$\lambda_n = \sum_{i=n+1}^{n+s(n)-1} \lambda_i + \sum_{i=n+s(n)+1}^{\infty} \varepsilon_i^*(\lambda_{n+s(n)})\lambda_i.$$

Thus using twice the biadditive property of F^* we have

$$a_n = \sum_{i=n+1}^{n+s(n)-1} a_i + \sum_{i=n+s(n)+1}^{\infty} \varepsilon_i^*(\lambda_{n+s(n)})a_i = \sum_{i=n+1}^{n+s(n)} a_i \leq \sum_{i=n+1}^{n_1} a_i,$$

where, by $a_n > 0$ there exists $n_1 := \max\{l \in \mathrm{N} \mid n+1 \leq l \leq n+s(n),\ l \in P_+\}$. That is (2.2) holds.

After this let $n \in P_+$ be such that the quasiregular expansion of λ_n is

$$\lambda_n = \sum_{i=n+1}^{n+s(n)-1} \lambda_i + \sum_{i=n+s(n)+1}^{\infty} \varepsilon_i^*(\lambda_n)\lambda_i, \tag{2.3}$$

where $s(n) \geq 3$ and $0 < \sum_{i=n+s(n)+1}^{\infty} \varepsilon_i^*(\lambda_n)\lambda_i < \lambda_{n+s(n)}$. By the continuity of F^* there exists a number $\xi_{n+s(n)}$ such that

$$\max_{x\in[0,L_{n+s(n)}]} F^*(x) =: F^*(\xi_{n+s(n)}),$$

and $F^*(\xi_{n+s(n)}) > 0$ because of the infinite property of P_+. Moreover by (2.1)

$$\lambda_n \geq L_{n+2} \tag{2.4}$$

holds for every $n \in \mathrm{N}$, therefore, by $s(n) \geq 3$

$$0 < \xi_{n+s(n)} \leq L_{n+s(n)} \leq \lambda_{n+s(n)-1} + \lambda_{n+s(n)} \leq \lambda_{n+s(n)-2} \leq \lambda_{n+1}. \tag{2.5}$$

Two cases are now possible:

$$I. \quad \sum_{i=n+s(n)+1}^{\infty} \varepsilon_i^*(\lambda_n)\lambda_i + \xi_{n+s(n)} \leq L_{n+s(n)}, \tag{2.6}$$

$$II. \quad \sum_{i=n+s(n)+1}^{\infty} \varepsilon_i^*(\lambda_n)\lambda_i + \xi_{n+s(n)} > L_{n+s(n)}. \tag{2.7}$$

In the first case let

$$x := \lambda_n + \xi_{n+s(n)}.$$

Then, by (2.3) and (2.6) $x \le L_{n+1} \le \lambda_{n-1}$ and on the other hand, by $s(n) \ge 3$

$$x > \sum_{i=n+1}^{n+s(n)-1} \lambda_i \ge \sum_{i=n+2}^{n+s(n)-2} \lambda_i + L_{n+s(n)+1}.$$

Thus, by (1.2), (1.3) and (2.5) the quasiregular expansion of x is

$$x = \lambda_n + \sum_{i=n+1}^{\infty} \varepsilon_i^*(x)\lambda_i = \lambda_n + \sum_{i=n+2}^{\infty} \varepsilon_i^*(\xi_{n+s(n)})\lambda_i,$$

and the antiregular expansion of x is

$$x = \sum_{i=n+1}^{n+s(n)-2} \lambda_i + \gamma\lambda_{n+s(n)-1} + \sum_{i=n+s(n)}^{\infty} \varepsilon_i^0(x)\lambda_i,$$

where $\gamma \in \{0,1\}$. Therefore, by the triadditivity of F^* we have

$$a_n + F^*(\xi_{n+s(n)}) = \sum_{i=n+1}^{n+s(n)-2} a_i + \gamma a_{n+s(n)-1} + F^*\Big(\sum_{i=n+s(n)}^{\infty} \varepsilon_i^0(x)\lambda_i\Big).$$

Then, by the definition of $\xi_{n+s(n)}$

$$a_n \le \sum_{i=n+1}^{n+s(n)-2} a_i + \gamma a_{n+s(n)-1} = \sum_{i=n+1}^{t} a_i,$$

where $t = n + s(n) - 2$ if $\gamma = 0$ and $t = n + s(n) - 1$ if $\gamma = 1$, that is, by $s(n) \ge 3, t > n$. So, by $a_n > 0$ there exists $n_1 \in \mathrm{N}$ such that $t \ge n_1 > n, n_1 \in P_+$ and (2.2) holds.

In the second case let

$$K_{n+s(n)} := L_{n+s(n)} - \sum_{i=n+s(n)+1}^{\infty} \varepsilon_i^*(\lambda_n)\lambda_i. \tag{2.8}$$

Then, by (2.3) $L_{n+s(n)+1} < K_{n+s(n)} < L_{n+s(n)}$. Let

$$\max_{x\in[0,K_{n+s(n)}]} F^*(x) =: F^*(\eta_{n+s(n)}).$$

Thus $0 < F^*(\eta_{n+s(n)}) \le F^*(\xi_{n+s(n)})$ and $0 < \eta_{n+s(n)} \le K_{n+s(n)} < \lambda_{n+1}$. Let

$$y := \lambda_n + \eta_{n+s(n)}.$$

Then $\lambda_n < y < \lambda_{n-1}$, so the quasiregular expansion of y is

$$y = \lambda_n + \sum_{i=n+2}^{\infty} \varepsilon_i^*(\eta_{n+s(n)})\lambda_i. \tag{2.9}$$

If

$$\lambda_{n+s(n)-1} + \sum_{i=n+s(n)+1}^{\infty} \varepsilon_i^*(\lambda_n)\lambda_i + \eta_{n+s(n)} > L_{n+s(n)} \tag{2.10}$$

then, by (2.3)

$$y = \sum_{i=n+1}^{n+s(n)-1} \lambda_i + \sum_{i=n+s(n)+1}^{\infty} \varepsilon_i^*(\lambda_n)\lambda_i + \eta_{n+s(n)} > \sum_{i=n+1}^{n+s(n)-2} \lambda_i + L_{n+s(n)}.$$

That is, by $y < L_{n+1}$ the antiregular expansion of y is

$$y = \sum_{i=n+1}^{n+s(n)-1} \lambda_i + \beta\lambda_{n+s(n)} + \sum_{i=n+s(n)+1}^{\infty} \varepsilon_i^0(\lambda_n)\lambda_i, \tag{2.11}$$

where $\beta \in \{0, 1\}$. Therefore, by (2.9), (2.11) and the triadditivity of F^* we have

$$a_n + F^*(\eta_{n+s(n)}) = \sum_{i=n+1}^{n+s(n)-1} a_i + \beta a_{n+s(n)} + F^*\Bigg(\sum_{i=n+s(n)+1}^{\infty} \varepsilon_i^0(y)\lambda_i\Bigg).$$

Thus, by the definition of $\eta_{n+s(n)}$

$$a_n \le \sum_{i=n+1}^{n+s(n)-1} a_i + \beta a_{n+s(n)} = \sum_{i=n+1}^{l} a_i,$$

where $l = n + s(n) - 1$ if $\beta = 0$ and $l = n + s(n)$ if $\beta = 1$. So, by $a_n > 0$ there exists $n_1 \in \mathrm{N}$ such that $l \ge n_1 > n, n_1 \in P_+$ and (2.2) holds.

If

$$\lambda_{n+s(n)-1} + \sum_{i=n+s(n)+1}^{\infty} \varepsilon_i^*(\lambda_n)\lambda_i + \eta_{n+s(n)} \le L_{n+s(n)}, \tag{2.12}$$

then, by (2.4) the antiregular expansion of y is

$$y = \sum_{i=n+1}^{n+s(n)-2} \lambda_i + \lambda_{n+s(n)} + \sum_{i=n+s(n)+1}^{\infty} \varepsilon_i^0(\lambda_n)\lambda_i.$$

Thus we have

$$a_n \leq \sum_{i=n+1}^{n+s(n)-2} a_i + a_{n+s(n)}. \tag{2.13}$$

Now we prove that $a_{n+s(n)-1} \geq 0$. By (2.5), (2.7), (2.12) we have

$$\lambda_{n+s(n)-1} + \lambda_{n+s(n)} \geq \xi_{n+s(n)} >$$

$$> L_{n+s(n)} - \sum_{i=n+s(n)+1}^{\infty} \varepsilon_i^*(\lambda_n)\lambda_i \geq \lambda_{n+s(n)-1} + \eta_{n+s(n)}.$$

Therefore, by $\eta_{n+s(n)} > 0$ the quasiregular expansion of $\xi_{n+s(n)}$ is the following:

$$\xi_{n+s(n)} = \lambda_{n+s(n)-1} + \sum_{i=n+s(n)+1}^{\infty} \varepsilon_i^*(\xi_{n+s(n)})\lambda_i.$$

Thus, by the triaddivity of F^*

$$F^*(\eta_{n+s(n)}) \leq F^*(\xi_{n+s(n)}) = a_{n+s(n)-1} + F^*\left(\sum_{i=n+s(n)+1}^{\infty} \varepsilon_i^*(\xi_{n+s(n)})a_i\right) \leq$$

$$\leq a_{n+s(n)-1} + F^*(\eta_{n+s(n)}),$$

that is, $a_{n+s(n)-1} \geq 0$. By these and by (2.13) and $a_n > 0$ we have that there exists $n_1 \in \mathrm{N}$ such that $n + s(n) \geq n_1 > n$, $n_1 \in P_+$ and (2.2) holds. Thus the proof of the Lemma 1 is complete.

Lemma 2. *For any $n \in P_+$*

$$a_n \leq \sum_{i=n+1}^{\infty} a_i. \tag{2.14}$$

Proof of the Lemma 2. Let $n \in P_+$ arbitrary. If $\lambda_n = L_{n+1}$ then by the triaddivity of F^*

$$a_n = \sum_{i=n+1}^{\infty} a_i$$

that is (2.14) holds. In the other case, if $\lambda_n \neq L_{n+1}$, by the Lemma 1 there exists $n_1 > n$ such that $n_1 \in P_+$ and (2.2) holds. If $\lambda_{n_1} \neq L_{n_1+1}$ then by the Lemma 1 there exists $n_2 > n_1$ such that $n_2 \in P_+$ and for $n = n_1$ (2.2) holds with the choice $n_1 = n_2$. Continuing this procedure, let us suppose that $n, n_1, n_2, \ldots, n_{t-1}$ are already defined ($n < n_1 < n_2 < \ldots < n_t, \{n, n_1, n_2, \ldots, n_t\} \subset P_+$) and $\lambda_{n_t} = L_{n_t+1}$. Then, by the triadditivity of F^* the inequalities (2.2) imply

$$a_n \leq a_n + a_{n_1} + a_{n_2} + \ldots + a_{n_t} \leq$$

$$\text{(2.15)} \qquad \leq \sum_{i=n+1}^{n_1} a_i + \sum_{i=n_1+1}^{n_2} a_i + \ldots + \sum_{i=n_{t-1}+1}^{n_t} a_i + \sum_{i=n_t+1}^{\infty} a_i = \sum_{i=n+1}^{\infty} a_i.$$

Otherwise $n < n_1 < n_2 < \ldots$ ($n, n_k \in P_+; k = 1, 2, \ldots$) are always defined, hence the inequalities (2.2) yield

$$a_n < a_n + a_{n_1} + a_{n_2} + \ldots \leq$$

$$\text{(2.16)} \qquad \leq \sum_{i=n+1}^{n_1} a_i + \sum_{i=n_1+1}^{n_2} a_i + \sum_{i=n_2+1}^{n_3} a_i + \ldots = \sum_{i=n+1}^{\infty} a_i.$$

By the inequalities (2.15) and (2.16), for any $n \in P_+$ we have

$$a_n \leq \sum_{i=n+1}^{\infty} a_i.$$

Thus we have proved the Lemma 2.

Using the lemmas we can prove the theorem. Let us now consider the function $-F^*$, also triadditive with respect to λ. Now

$$P_- := \{n \in \mathrm{N} \mid a_n < 0\} = \{n \mid -a_n > 0\},$$

hence by the lemmas, for any $n \in P_-$ we have

$$\text{(2.17)} \qquad -a_n \leq \sum_{i=n+1}^{\infty} (-a_i).$$

Now by the infinite property of P_+ and of P_- there exist $n \in P_+$ and $k \geq 0$ such that $n + k + 1 \in P_-$ and if $k \geq 1$ then $\{n+1, n+2, \ldots, n+k\} \subset P_0$. Hence by (2.14) and (2.17) we obtain

$$a_n \leq \sum_{i=n+1}^{\infty} a_i = \sum_{i=n+k+1}^{\infty} a_i$$

and

$$-a_{n+k+1} \leq \sum_{i=n+k+2}^{\infty} (-a_i).$$

Adding the two previous inequalities we have

$$a_n \leq 2a_{n+k+1} < 0,$$

and this contradicts the inequality $a_n > 0$ $(n \in P_+)$. Thus we have proved that $F^* \equiv 0$, that is

$$F(x) = \frac{F(L(\lambda))}{L(\lambda)} x$$

for every $x \in [0, L(\lambda)]$.

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(Received May 9, 1990)

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