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SOLUTION OF SYSTEM OF HIGH-ORDER DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS VIA BLOCK-PULSE FUNCTIONS

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1. Introduction

Approximating a function as a linear combination of a set of orthogonal basis functions is a standard tool in a numerical analysis. Corrington [1] proposed a method of solving nonlinear differential and integral equations using a set of Walsh functions as the basis. His method is aimed at obtaining piecewise constant (approximate) solutions of dynamic equations, and requires previously prepared tables of coefficients for integrating Walsh functions. To alleviate the need for such tables, Chen and Hsiao [2] introduced an operational matrix to perform integration of Walsh functions.

This paper simplifies and is the generalization to the method of Chen and Hsiao [2] by using block-pulse functions instead of Walsh functions as the basis for solving a system of high-order differential equations with constant coefficients simultaneously. First the block-pulse functions are introduced and their properties briefly summarized [3], [4]. Then the block-pulse function series of t^k, $0 \le t < 1$, $k \in \mathbb{N} = \{0, 1, \ldots\}$, is established. The operational matrix for block-pulse functions is introduced in [4] and a system of high-order differential equations with constant coefficients is solved simultaneously by using the block-pulse functions. Finally, depending on the special properties of the operational matrix, we give a simple method to compute the solution of the equation obtained in (29) (see Appendix).

2. Block-pulse functions (b.p.f.)

A set of b.p.f. on the unit interval $[0, 1)$ is defined as follows [4]: for each integer i, $0 \le i < m$ and $m \in \mathbb{P} = \{1, 2, \ldots\}$ the function ϕ_i is given by

$$(1) \qquad \phi_i(t) = \begin{cases} 1 & \text{for } \dfrac{i}{m} \le t < \dfrac{i+1}{m}, \\ 0 & \text{otherwise.} \end{cases}$$

This set of functions can be concisely described by an m-vector $\Phi_{(m)}$ with ϕ_i as its i-th component. It is well-known that a function f which is integrable in $[0,1]$ can be approximated as

$$f \simeq \sum_{i=0}^{m-1} a_i \phi_i, \tag{2}$$

where the coefficients a_i are such that

$$\int_0^1 \left[f(t) - \sum_{i=0}^{m-1} a_i \phi_i(t) \right]^2 dt$$

is minimized. In fact, a_i is given by

$$a_i = m \int_{\frac{i}{m}}^{\frac{i+1}{m}} f(t)dt, \quad 0 \le i < m. \tag{3}$$

The b.p.f. satisfies the properties

$$\phi_i \phi_j = \delta_{ij} \phi_i \tag{4}$$

and

$$\int_0^1 \phi_i(t)\phi_j(t)dt = \frac{1}{m}\delta_{ij}, \tag{5}$$

where δ_{ij} denotes the Kronecker δ symbol.

3. B.p.f. series for t^k, $0 \le t < 1$, $k \in \mathbb{N}$ and the operational matrix of integration of b.p.f.

The function t^k, $t \in [0,1)$, $k \in \mathbb{N}$ can be approximated as a b.p.f. series of size m. Indeed, from (2) and (3), we have

$$t^k \simeq \sum_{i=0}^{m-1} a_k(i)\phi_i(t), \tag{6}$$

where

$$(7)\qquad a_k(i) = m \int\limits_{\frac{i}{m}}^{\frac{i+1}{m}} t^k dt = \frac{m}{k+1}\left[\left(\frac{i+1}{m}\right)^{k+1} - \left(\frac{i}{m}\right)^{k+1}\right].$$

Therefore,

$$(8)\qquad t^k \simeq \frac{m}{k+1}\sum_{i=0}^{m-1}\left[\left(\frac{i+1}{m}\right)^{k+1} - \left(\frac{i}{m}\right)^{k+1}\right]\phi_i(t)$$

and in matrix form

$$(9)\qquad t^k \simeq \frac{m}{k+1} Y_k^T \Phi_{(m)}(t),$$

where

$$(10)\qquad Y_k^T = \left(\left(\frac{1}{m}\right)^{k+1}, \left(\frac{2}{m}\right)^{k+1} - \left(\frac{1}{m}\right)^{k+1}, \ldots, 1 - \left(\frac{m-1}{m}\right)^{k+1}\right).$$

The first integral of b.p.f. can be expressed by b.p.f. Indeed, from (1) we have

$$(11)\qquad \int\limits_0^t \phi_i(\lambda)d\lambda = \begin{cases} 0, & 0 \le t < \frac{i}{m}, \\ t - \frac{i}{m}, & \frac{i}{m} \le t < \frac{i+1}{m}, \\ \frac{1}{m}, & \frac{i+1}{m} \le t < 1. \end{cases}$$

The four-interval b.p.f. and their integrals are shown in Fig. 1. Then (11) can be written as

$$(12)\qquad \int\limits_0^t \phi_i(\lambda)d\lambda = \left(t - \frac{i}{m}\right)\phi_i(t) + \frac{1}{m}\sum_{j=i+1}^{m-1}\phi_j(t).$$

From (8) we have

$$(13)\qquad t \simeq \frac{m}{2}\sum_{i=0}^{m-1}\left[\left(\frac{i+1}{m}\right)^2 - \left(\frac{i}{m}\right)^2\right]\phi_i(t).$$

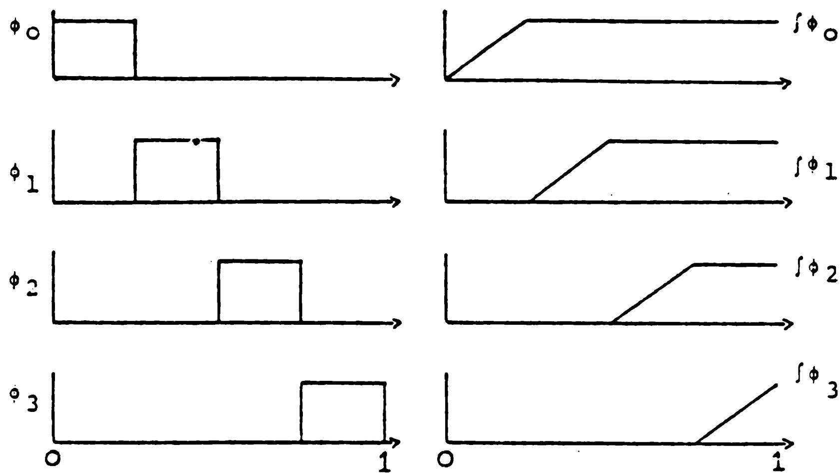

Fig.1. B.p.f. and their integrations

Substituting (13) and (4) into (12), we have for $0 \leq i < m$

$$\int_0^t \phi_i(\lambda)d\lambda =$$

$$= \frac{m}{2}\sum_{j=0}^{n-1}\left[\left(\frac{j+1}{m}\right)^2 - \left(\frac{j}{m}\right)^2\right]\phi_j(t)\phi_i(t) - \frac{i}{m}\phi_i(t) + \frac{1}{m}\sum_{j=i+1}^{m-1}\phi_j(t) =$$

$$= \frac{m}{2}\left[\left(\frac{i+1}{m}\right)^2 - \left(\frac{i}{m}\right)^2\right]\phi_i(t) - \frac{i}{m}\phi_i(t) + \frac{1}{m}\sum_{j=i+1}^{m-1}\phi_j(t) =$$

$$= \frac{1}{2m}\phi_i(t) + \frac{1}{m}\sum_{j=i+1}^{m-1}\phi_j(t).$$

Therefore, we can write the relationship between b.p.f. and their integrals in the matrix form

$$(14)\qquad \begin{bmatrix} \int\phi_0 \\ \int\phi_1 \\ \vdots \\ \int\phi_{m-1} \end{bmatrix} \simeq \frac{1}{m} \begin{bmatrix} \frac{1}{2} & 1 & 1 & 1 & & 1 \\ 0 & \frac{1}{2} & 1 & 1 & & 1 \\ 0 & 0 & \frac{1}{2} & 1 & & 1 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \phi_0 \\ \phi_1 \\ \vdots \\ \phi_{m-1} \end{bmatrix}$$

or in more compact form

$$(15)\qquad \int_0^t \Phi_{(m)}(\lambda)d\lambda \simeq B\Phi_{(m)}(t).$$

B is called the operational matrix which relates the b.p.f. and their integrals. The operational matrix B is a triangular matrix and it has some properties which reduce the calculations in solving a system of differential equations.

4. Solutions of system of high-order differential equations with constant coefficients

Consider the following system of differential equations of order r ($r \geq 1$) with constant coefficients

$$(16)\qquad X^{(r)} + \sum_{k=1}^{r} A_{r-k}X^{(r-k)} = EU, \quad X^{(i)}(0) = X_0^{(i)} \quad (i = 0, 1, \ldots, r-1),$$

where X is a vector of n components, U is an input vector of ℓ components, A_i ($i = 0, 1, \ldots, r-1$) are $n \times n$ matrices and E is $n \times \ell$ matrix. For solving this problem by the b.p.f., we expand $X^{(r)}$, U in b.p.f. series of size m

$$(17)\qquad X^{(r)} \simeq \sum_{i=0}^{m-1} c_i\phi_i = C\Phi_{(m)},$$

$$(18)\qquad U \simeq H\Phi_{(m)},$$

where c_i is n-vector and forms the i-th column of the $n \times m$ matrix C and H is $\ell \times m$ matrix. The elements of the matrix C are yet unknown, whereas the elements of H can be obtained for a given input U by applying (3).

Now integrating (17) from 0 to t and using (15) we get

$$X^{(r-1)} = CB\Phi_{(m)} + X_0^{(r-1)}. \tag{19}$$

In fact, the k-th integration of (17) yields

$$X^{(r-k)}(t) = CB^k\Phi_{(m)}(t) + \sum_{i=1}^{k} X_0^{(r-i)} \frac{t^{k-i}}{(k-i)!} \qquad (k = 1, 2, \ldots, r), \tag{20}$$

$(t \in [0, 1))$.
From (9) and (10) we have

$$\frac{t^{k-i}}{(k-i)!} \simeq \frac{m}{(k-i+1)!} Y_{k-i}^T \Phi_{(m)}(t). \tag{21}$$

Substituting (21) into (20) we get

$$X^{(r-k)} = CB^k\Phi_{(m)} + Z_k\Phi_{(m)}, \tag{22}$$

where

$$Z_k = m\sum_{i=1}^{k} \frac{m}{(k-i+1)!} X_0^{(r-i)} Y_{k-i}^T \tag{23}$$

is an $n \times m$ constant matrix.

Substituting (17), (18) and (22) into (16), we have

$$C\Phi_{(m)} + \sum_{k=1}^{r} A_{r-k} \left(CB^k + Z_k\right) \Phi_{(m)} = EH\Phi_{(m)}. \tag{24}$$

Therefore,

$$C = -\sum_{k=1}^{r} A_{r-k} CB^k + V, \tag{25}$$

where

$$V = EH - \sum_{k=1}^{r} A_{r-k} Z_k \tag{26}$$

is an $n \times m$ matrix.

Let $v_0, v_1, \ldots, v_{m-1}$ be the columns of V, then (25) is expressible as follows

$$(27) \qquad [c_0, c_1, \ldots, c_{m-1}] = -\sum_{k=1}^{r} A_{r-k}[c_0, c_1, \ldots, c_{m-1}]B^k + [v_0, v_1, \ldots, v_{m-1}].$$

Using the Kronecker product technique introduced by Chen and Hsiao [2], we rearrange C as a vector with nm components by changing its first column into the first n components of the vector and then the second column, etc.; and rearrange V in the same manner, finally we obtain

$$(28) \qquad \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{m-1} \end{bmatrix} = -\sum_{k=1}^{r} [A_{r-k} \otimes (B^k)^T] \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{m-1} \end{bmatrix} + \begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ v_{m-1} \end{bmatrix},$$

where $A_{r-k} \otimes (B^k)^T$ is the Kronecker product of the two matrices (see Appendix). The solution of C comes from (28) directly,

$$(29) \qquad \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{m-1} \end{bmatrix} = \left[I + \sum_{k=1}^{r} A_{r-k} \otimes (B^k)^T \right]^{-1} \begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ v_{m-1} \end{bmatrix},$$

where I is the $nm \times nm$ identity matrix.

After C is determined the solution X is easily found by substituting C into (22), namely

$$(30) \qquad X = CB^r \Phi_{(m)} + Z_r \Phi_{(m)}.$$

A question arises now. "How many terms m of b.p.f. should we use?" If we wish to obtain a quick answer and to sacrifice accuracy, we can use small number for m, say, let $m \leq 8$. On the other hand, if we want accurate answer and do not care about computation time, we can use very large value for m, say, $m \geq 128$. We notice that, if we use (29) directly to solve a system of six differential equations via the b.p.f. of size $m = 128$, some difficulties might occur in obtaining the inverse of a square matrix of 768×768.

According to the properties of the operational matrix B, a simple method is established to solve C from (29).

Appendix

In this appendix we give a simple method to find the inverse of the matrix $\left[I + \sum_{k=1}^{r} A_{r-k} \otimes (B^k)^T\right]$ depending on the special properties of matrix B and then the solution C from (29) is easily obtained.

First we note that the matrix B is triangular and has the form

$$B = \frac{1}{2m}\begin{bmatrix} b_1^{(1)} & b_2^{(1)} & b_3^{(1)} & b_4^{(1)} & & b_m^{(1)} \\ 0 & b_1^{(1)} & b_2^{(1)} & b_3^{(1)} & \dots & b_{m-1}^{(1)} \\ 0 & 0 & b_1^{(1)} & b_2^{(1)} & \dots & b_{m-2}^{(1)} \\ \vdots & & & & & \\ 0 & 0 & 0 & 0 & & b_1^{(1)} \end{bmatrix}, \tag{31}$$

where

$$b_1^{(1)} = 1 \quad \text{and} \quad b_i^{(1)} = 2 \quad (i = 2, 3, \dots, m).$$

An elementary calculation shows that the powers B^k $(k > 1)$ has the form

$$B^k = \frac{1}{(2m)^k}\begin{bmatrix} b_1^{(k)} & b_2^{(k)} & b_3^{(k)} & b_4^{(k)} & & b_m^{(k)} \\ 0 & b_1^{(k)} & b_2^{(k)} & b_3^{(k)} & \dots & b_{m-1}^{(k)} \\ 0 & 0 & b_1^{(k)} & b_2^{(k)} & & b_{m-2}^{(k)} \\ \vdots & & & & & \\ 0 & 0 & 0 & 0 & & b_1^{(k)} \end{bmatrix}, \tag{32}$$

where the elements of B^k are given by the following recursive formulae:

$$\begin{aligned} & b_1^{(j)} = 1, \\ & b_i^{(j)} = \sum_{s=1}^{i} b_s^{(j-1)} b_{i-s+1}^{(1)} \qquad (j = 2, 3, \dots, k;\ i = 2, 3, \dots, m). \end{aligned} \tag{33}$$

Consequently,

$$A_{r-k} \otimes (B^k)^T =$$

(34)
$$= \frac{1}{(2m)^k} \begin{bmatrix} A_{r-k} & 0 & 0 & 0 \\ b_2^{(k)} A_{r-k} & A_{r-k} & 0 & 0 \\ b_3^{(k)} A_{r-k} & b_2^{(k)} A_{r-k} & A_{r-k} & 0 \\ & & & \\ b_m^{(k)} A_{r-k} & b_{m-1}^{(k)} A_{r-k} & b_{m-2}^{(k)} A_{r-k} \quad \dots & A_{r-k} \end{bmatrix}.$$

Therefore,

(35)
$$\left[I + \sum_{k=1}^{r} A_{r-k} \otimes (B^k)^T \right] = \begin{bmatrix} P_1 & 0 & 0 & 0 \\ P_2 & P_1 & 0 & 0 \\ P_3 & P_2 & P_1 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ P_m & P_{m-1} & P_{m-2} & P_1 \end{bmatrix},$$

where

(36)
$$P_1 = I_{(n \times n)} + \sum_{k=1}^{r} \frac{1}{(2m)^k} A_{r-k}$$

and

(37)
$$P_i = \sum_{k=1}^{r} \frac{b_i^{(k)}}{(2m)^k} A_{r-k} \qquad (i = 2, 3, \dots, m)$$

are $n \times n$ matrices.

Now a simple form for the inverse of the matrix in (35) is obtained. Indeed

(38)
$$\left[I + \sum_{k=1}^{r} A_{r-k} \otimes (B^k)^T \right]^{-1} = \begin{bmatrix} R_1 & 0 & 0 & 0 \\ R_2 & R_1 & 0 & 0 \\ R_3 & R_2 & R_1 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ R_m & R_{m-1} & R_{m-2} \quad \dots & R_1 \end{bmatrix},$$

where R_i, $i = 1, 2, \ldots, m$ are $n \times n$ matrices determined by the following recursive formulae

$$R_1 = P_1^{-1},$$

(39)
$$R_i = -\sum_{j=1}^{i-1} R_1 P_{i-j+1} R_j \qquad (i = 2, 3, \ldots, m).$$

Substituting (38) into (29), the solution C is easily obtained

$$c_j = \sum_{i=0}^{j} R_{j-i+1} v_i \qquad (j = 0, 1, \ldots, m-1).$$

In the above method we always work with matrices (P, R) of $n \times n$, and the inverse of only one $n \times n$ matrix, P_1 is computed. Therefore, we have saved computing time and storage. In addition we have reduced round-off errors significantly.

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