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ON FUZZY SET OF SETS OF SMALL MEASURE

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Abstract: In many problems of measure theory it is not necessary to know the value $\mu(E)$ of the measure of a set E, but only whether $\mu(E)=0$ or not. In other problems a similar role the set of "small measure" plays. Of course, this is a fuzzy notion. The aim of the article is to present two models of the phenomenon and compare them.

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1. SMALL SYSTEMS OF SETS

In [7] the following scheme was suggested for the modeling the sets of small measure. A non-empty set X and a σ-ring S of subsets of X (i.e. a family of subsets of X closed with respect to the countable union and the difference of two sets) are considered. The smallness is expressed by a sequence $(\mathcal{N}_n)_n$ of subfamilies of S. The membership to $\mathcal{N}_n$ means that the considered set is small in the n-th degree. The classical example is the sequence defined by

$$\mathcal{N}_n = \{E \in S; \mu(E) < \frac{1}{n}\}.$$

Definition 1. Let (X, S) be a measurable space. A sequence $(\mathcal{N}_n)_n \subset S$ is called to be a small system, if the following properties are satisfied:

(i) $\emptyset \in \mathcal{N}_{n+1} \subset \mathcal{N}_n$ for all $n \in N$.

(ii) If $A \in \mathcal{N}_n, B \subset A, B \in S$, then $B \in \mathcal{N}_n$.

(iii) For every $n \in N$ there are $k_i \in N (i = 1, 2, \dots)$ such that $E_i \in \mathcal{N}_{k_i}$ $(i = 1, 2, \dots)$ implies $\bigcup_{i=1}^{\infty} E_i \in \mathcal{N}_n$.

(iv) If $E_i \in \mathcal{S}, E_i \supset E_{i+1} (i = 1, 2, \ldots)$ and $\bigcap_{i=1}^{\infty} E_i = \emptyset$, then for every $n \in N$ there is $i \in N$ such that $E_i \in \mathcal{N}_n$. □

There are at least 20 papers related to the small systems, we list only some typical of them in References. Further we present here a typical result - the regularity theorem for Baire measures (in a very special case).

Proposition 2. *Let $\mathcal{S} = \mathcal{B}(\mathbb{R}^n)$ be the set of all Borel subsets of the n-dimensional space $\mathbb{R}^n$. Then for every $E \in \mathcal{S}$ and every $n \in N$ there are an open set U and a closed set C such that $C \subset E \subset U$ and $U \setminus C \in \mathcal{N}_n$.* □

Proof. See [7]. ■

As corollaries of Proposition 2 one can obtain not only the classical Baire measure regularity theorem but also corresponding assertions for subadditive measures as well as for vector measures in Banach spaces and in some type of Riesz spaces.

2. FUZZY SET OF SMALL SETS

Definition 3. Let $(X, \mathcal{S})$ be a measurable space. By a fuzzy set of small sets we mean any mapping $m : \mathcal{S} \to [0, 1]$ satisfying the following conditions:

(i) $m(\emptyset) = 1$.

(ii) If $A \subset \bigcup_{i=1}^{\infty} A_i, A \in \mathcal{S}, A_i \in \mathcal{S} (i = 1, 2, \ldots)$, then $m(A) \geq \bigcup_{i=1}^{\infty} m(A_i)$.

(iii) If $A_i \in \mathcal{S},\ A_i \supset A_{i+1},\ (i = 1, 2, \ldots)$ and $\bigcap_{i=1}^{\infty} A_i = \emptyset$, then $\lim_{i \to \infty} m(A_i) = 1$. □

Definition 4. A sequence $(\mathcal{N}_n)_n \subset \mathcal{S}$ and a function $m : \mathcal{S} \to [0, 1]$ are called to be equivalent if the following conditions hold:

(i) $\forall \varepsilon > 0\ \exists n \in N : A \in \mathcal{N}_n \Rightarrow m(A) > 1 - \varepsilon$.

(ii) $\forall n \in N\ \exists \varepsilon > 0 : m(A) > 1 - \varepsilon \Rightarrow A \in \mathcal{N}_n$. □

Theorem 5. *For every fuzzy set $m : \mathcal{S} \to [0, 1]$ of small sets there exists a small system $(\mathcal{N}_n)_n \subset \mathcal{S}$ equivalent with m.* □

Proof. Put

$$\mathcal{N}_n = \{E \in \mathcal{S}; m(E) > e^{-1/n}\}.$$

Then all claimed assertions are easy to be proved.
More interesting is the construction in the opposite direction. ■

Theorem 6. *For every small system $(\mathcal{N}_n)_n \subset \mathcal{S}$ there exists a fuzzy set $m : \mathcal{S} \to [0,1]$ of small sets equivalent with $(\mathcal{N}_n)_n$.* □

Proof. Put

$$\begin{aligned} h(E) &= \sup\{n \in N;\ E \in \mathcal{N}_n\}, \\ f(E) &= e^{-h(E)}, \\ p(E) &= \inf\{\sum_{i=1}^{n} f(E_i);\ E = \bigcup_{i=1}^{n} E_i,\ E_i \in \mathcal{S},\ n \in N\}, \\ m(E) &= e^{-p(E)}. \end{aligned}$$

Evidently $h(\emptyset) = \infty$, $p(\emptyset) = f(\emptyset) = 0$, hence $m(\emptyset) = 1$. By [3] p is a monotone, σ-subadditive, continuous (i.e. $A_i \searrow \emptyset \Rightarrow p(A_i) \searrow 0$) and

$$p(A) \le f(A) \le 2p(A) \tag{$*$}$$

so the properties (ii) and (iii) hold.
We shall prove the equivalence. Let $0 < \varepsilon < 1$. Choose $n \in N$ such that $n > -\ln(-\ln(1-\varepsilon))$, i.e. $\exp(-e^{-n}) > 1-\varepsilon$. If $E \in \mathcal{N}_n$, then $h(E) \ge n$, $f(E) = e^{-h(E)} \le e^{-n}$, hence by $(*)$ $p(E) \le f(E) \le e^{-n}$. Therefore $m(E) = \exp(-e^{-n}) > 1-\varepsilon$.
Contrary, for a given $\varepsilon > 0$ let choose n such that $\varepsilon < 1 - \exp(-\frac{1}{n}e^{-n})$. i.e. $-\ln(-2\ln(1-\varepsilon)) > n$. If $m(E) > 1-\varepsilon$, then $p(E) = -\ln(m(E)) < -\ln(1-\varepsilon)$. Of course, by $(*)$

$$e^{-h(E)} = f(E) \le 2p(E) < -2\ln(1-\varepsilon),$$

$$h(E) > -\ln(-2\ln(1-\varepsilon)) > n,$$

i.e. $E \in \mathcal{N}_n$. ■

Recall finally that there exists also a lattice variant of theorems 5 and 6 (see [4]) with more complicated formulation and another proof. Of course, the lattice variant includes our results, but the present formulations seem to be more convenient for applications.

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