

## FUZZY LINEAR SYSTEMS

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**Abstract:** The resolution problem for fuzzy linear systems of equations and inequalities over a bounded chain is studied. The main results are concerned with compatibility, computing solutions or marking the contradictory equations (resp. inequalities).

**Keywords:** Fuzzy linear systems, fuzzy equations, fuzzy inequalities.

### 1. PRELIMINARIES

The resolution problem for fuzzy relation equations has been set by E. Sanchez [5]. Following his fundamental result for the greatest solution, many other authors proposed thoroughly investigations for a variety of special resolution problems [1,2,3].

The purpose of this paper is to give a unified approach to the resolution problem for fuzzy linear systems of equations and inequalities over the bounded chain. We consider the problem of solving fuzzy linear systems of equations  $A \cdot X = B$  and inequalities  $A \cdot X \geq B$ ,  $A \cdot X > B$ ,  $A \cdot X < B$  in  $\mathbb{L}$ . Polynomial time algorithms are proposed for computing:

- (i) is the system consistent or not;
- (ii) if the system is consistent - when is the solution unique; what are its greatest, lower and maximal solutions;
- (iii) if the system is inconsistent - which are the numbers of the contradictory equations (resp. inequalities).

The complete text with all proofs, examples and applications is subject to [4].

Let  $\mathbb{L} = (L, \vee, \wedge, 0, 1)$  be a bounded chain over the totally ordered set  $L$  with universal bounds 0 and 1, with operations join  $\vee$  and meet  $\wedge$ . Let  $I, J \neq \emptyset$  be finite sets of indices.  $a : I \times J \rightarrow L$  be a map and  $\mathbb{L}^{I \times J} = \{a | a : I \times J \rightarrow L\}$  be the set of all maps from  $I \times J$  to  $L$ ;  $A = (a_{ij}) \in \mathbb{L}^{I \times J}$  is called a *matrix over*  $L$  if  $a_{ij} = a(i, j)$  for each  $i \in I$  and each  $j \in J$ .

The matrix  $C = A \cdot B = (c_{ij}) \in \mathbb{L}^{I \times J}$  such that

$$c_{ij} = \bigvee_{k=1}^{|K|} (a_{ik} \wedge b_{kj}) \text{ for each } i \in I, j \in J.$$

is called *product* of  $A = (a_{ij}) \in \mathbb{L}^{I \times K}$  and  $B = (b_{ij}) \in \mathbb{L}^{K \times J}$ .

The finite matrices  $A$ ,  $X$  and  $B$  denote coefficients, unknowns and constants respectively for the system under study.  $\{1\}$  stands for the singleton set,  $|I| = m \in \mathbb{N}$ ,  $|J| = n \in \mathbb{N}$  denote the cardinality of  $I$  and  $J$  respectively.

The system of linear equations is written more briefly as

$$\forall_{j \in J} (a_{ij} \wedge x_j) = b_i, \quad i \in I \quad (1)$$

We shall write shortly  $A \cdot X \geq B$ ,  $A \cdot X > B$ ,  $A \cdot X \leq B$ ,  $A \cdot X < B$  respectively for the systems (2), (3), (4), (5) below:

$$\forall_{j \in J} (a_{ij} \wedge x_j) \geq b_i, \quad i \in I \quad (2)$$

$$\forall_{j \in J} (a_{ij} \wedge x_j) > b_i, \quad i \in I \quad (3)$$

$$\forall_{j \in J} (a_{ij} \wedge x_j) \leq b_i, \quad i \in I \quad (4)$$

$$\forall_{j \in J} (a_{ij} \wedge x_j) < b_i, \quad i \in I \quad (5)$$

where  $A = (a_{ij} \in \mathbb{L}^{I \times J})$ ,  $X = (x_j) \in \mathbb{L}^{J \times \{1\}}$ ,  $B = (b_i) \in \mathbb{L}^{I \times \{1\}}$  and for all of them we suppose  $b_1 \geq \dots \geq b_m$ . If the notions, definitions or results are valid for any of them, we write  $A \cdot X \perp B$ .

Let the system  $A \cdot X \perp B$  be given. The row-matrix  $X^0 = (x_j^0) \in \mathbb{L}^{J \times \{1\}}$  is a *point solution* of  $A \cdot X \perp B$  if  $A \cdot X^0 \perp B$  holds. The set of all point solutions of  $A \cdot X \perp B$  is denoted by  $K^0$ . If  $K^0 \neq \emptyset$  then  $A \cdot X \perp B$  is called *consistent*, otherwise it is *inconsistent*.  $\underline{X}^0 \in K^0$  is called *lower point solution* of  $A \cdot X \perp B$  if for any  $X^0 \in K^0$  the relation  $X^0 \leq \underline{X}^0$  implies  $X^0 = \underline{X}^0$ , where  $\leq$  denotes the order, induced in  $K^0$  by this of  $\mathbb{L}$ . Dually,  $\overline{X}^0 \in K^0$  is an *upper point solution*, if for any  $X^0 \in K^0$  the relation  $\overline{X}^0 \leq X^0$  implies  $\overline{X}^0 = X^0$ . An interval  $X_j \subseteq L$  is called *feasible* for the  $j^{\text{th}}$  component of the solution, if the choice of any  $x_j \in X_j$  does not result in a contradiction in the system. An  $n$ -tuple  $(X_1, \dots, X_n)$  of feasible intervals  $X_j \subseteq L$  is called an *interval solution* if any  $X^0 = (x_1^0, \dots, x_n^0)$ , such that  $x_j^0 \in X_j$  ( $j = 1, \dots, n$ ) belongs to  $K^0$ . The interval solution, which is maximal with respect to this property, is called *maximal interval solution*.

Two systems are called *equivalent* if each solution of the first one is a solution of the second and vice versa.

We assign to the system  $A \cdot X \perp B$  a new system  $A^* \cdot X \perp B$  with an *augmented* matrix  $(A^* : B)$ . The matrix  $A^* = (a_{ij}^*)$  is carried out from  $A$  with respect to  $B$  according to (6):

$$a_{ij}^* = \begin{cases} S, & \text{if } a_{ij} < b_i; \\ E, & \text{if } a_{ij} = b_i; \\ G, & \text{if } a_{ij} > b_i. \end{cases} \quad (6)$$

**Lemma 1.** *The systems  $A \cdot X \perp B$  and  $A^* \cdot X \perp B$  are equivalent.* □

Let the system  $A \cdot X \perp B$  be given. The time complexity function for obtaining the equivalent system with  $b_1 \geq \dots \geq b_m$  is  $O(m^2)$ ; The time complexity function for computing  $(A^* : B)$  is  $O(m \cdot n)$ .

## 2. FUZZY LINEAR SYSTEMS IN $\mathbb{L}$

An unified method is presented for solving  $A \cdot X \perp B$  in  $\mathbb{L}$ , resulting in: the necessary and sufficient condition for consistency of any linear system in  $\mathbb{L}$  as an analogue of the Kronecker - Kappely Theorem; the greatest and lower solutions; a polynomial time algorithm for computing: is the system consistent or not; if it is inconsistent - why; if it is consistent - all its solutions.

Whenever  $A \cdot X \perp B$  is given, we shall suppose  $(A^* : B)$  is computed. According to Lemma 1 we study  $A^* \cdot X \perp B$  instead of  $A \cdot X \perp B$ . We assume the requirement  $b_1 \geq \dots \geq b_m$  is satisfied; we denote by  $k$  the greatest number of the row with  $G$ -type coefficient in it and by  $r$  the smallest number of the row with  $E$ -type coefficient in it.

**Theorem 1.** *Let the system  $A \cdot X = B$  be given*

i) *If the  $j^{\text{th}}$  column of  $A^*$  contains  $a_{kj}^* = G$ , then:*

a)  $X_j = [0, b_k]$  *is a feasible interval for the  $j^{\text{th}}$  component;*

b)  $x_j = b_k$  *implies  $a_{ij} \vee x_j = b_i$  for  $i = k$ , for each  $i < k$  with  $a_{ij} > b_i = b_k$  and for each  $i > k$  with  $a_{ij} = b_i$ ;*

ii) *If the  $j^{\text{th}}$  column of  $A^*$  does not contain any  $G$ -type coefficient, but it contains  $E$ -type coefficient  $a_{rj}^* = b_r$ , then:*

a)  $X_j = L$  *is the feasible interval for the  $j^{\text{th}}$  component;*

b)  $x_j \in [b_r, 1]$  *implies  $a_{ij} \wedge x_j = b_i$  for each  $i \geq r$  with  $a_{ij} = b_i$ ;*

iii) *If the  $j^{\text{th}}$  column of  $A^*$  does not contain neither  $G$ -type, nor  $E$ -type coefficients then the feasible interval is  $X_j = L$  and  $A_{ij} \wedge x_j < b_i$  holds for any  $x_j \in L$ . □*

We denote by  $\mathcal{G}$ ,  $\mathcal{IE}$  respectively all  $G$  or  $E$  coefficients selected to satisfy the  $i^{\text{th}}$  equation by the term  $a_{ij} \wedge x_j = b_i$  due to Theorem 1 (i), (ii). Those  $\mathcal{G}$  and  $\mathcal{IE}$  coefficients are *selected*.

**Corollary 1.** *Let the system  $A \cdot X = B$  be given.*

- i) *It is consistent iff there exists at least one selected coefficient  $A_i^* \in \{\mathbb{G}, \mathbb{IE}\}$  for each  $i \in I$ . If there exists an equation with no selected coefficient, then this equation is in contradiction with the others;*
- ii) *The time complexity function for establishing the consistency of the system is  $O(n \cdot m^2)$ .* □

**Corollary 2.** *Let the system  $A \cdot X = B$  be consistent. Then it has unique greatest solution  $X_{gr} = (x_j)_{n \times 1}$ , where:*

$$x_j = \begin{cases} b_k, & \text{if the } j^{\text{th}} \text{ column of } A^* \text{ contains } \mathbb{G}\text{-type} \\ & \text{coefficient due to Theorem 1 (i)(b);} \\ 1 & \text{otherwise.} \end{cases}$$

*The time complexity function for computing  $X_{gr}$  is  $O(m \cdot n)$ .* □

Let  $A \cdot X = B$  be consistent. It means that in any equation

$$(a_{i1} \wedge x_1) \vee \dots \vee (a_{in} \wedge x_n) = b_i, \quad i \in I$$

there are only terms with  $a_{ij} \wedge x_j \leq b_i$  and the equality  $a_{ij} \wedge x_j = b_i$  holds only for  $\mathbb{G}$  and  $\mathbb{IE}$  coefficients in this equation. Let  $J_i = \{j | j \in J, a_{ij} \wedge x_j = b_i\}$ ,  $i = 1, \dots, m$  stand for the set of the second indices of all  $\mathbb{IE}$  and  $\mathbb{G}$  coefficients in the  $i^{\text{th}}$  equation. We define the *help matrix*  $H = (h_{ij})$ :

$$h_{ij} = \begin{cases} b_i & \text{if } a_{ij} \wedge x_j = b_i \text{ holds for } j \in J_i \\ 0 & \text{otherwise.} \end{cases}$$

**Corollary 3.** *For any linear system  $A \cdot X = B$*

- i)  *$(A : B)$  and  $(A^* : B)$  have the same  $H$ ;*
- ii) *the time complexity function for computing  $H$  is  $O(m \cdot n)$ .* □

Using  $H$ , we can compute all the lower and maximal solutions [4].

**Corollary 4.** *If the system  $A \cdot X = B$  is consistent, then the set of all its lower solutions is finite and computable.* □

**Algorithm 1** (for solving the system  $(A \cdot X = B)$ ).

1. Enter the matrices  $A_{m \times n}$ ,  $B_{m \times 1}$  ( $|I| = m$ ,  $|J| = n$ ).
2. Compute the matrix  $(A^* : B)$ .
3. Erase the vector  $IND$  and the help-matrix  $H = (h_{ij})_{m \times n}$ .
4.  $X_{gr}(i) = 1$  and  $X_{low}(i) = 0$ ,  $i = 1, \dots, n$ .

5.  $j = 0$ .
6.  $j = j + 1$
7. If  $j > n$  go to 10.
8. If the  $j^{\text{th}}$  column in  $A^*$  does not contain any G-type coefficient, go to 9. Otherwise take the greatest number  $k$  of the row with G-type coefficient in the  $j^{\text{th}}$  column of  $A^*$ . Put  $X_{gr}(j) = b_k$ ,  $IND(i) = IND(i) + 1$  and  $h_{ij} = b_i$  for  $i > k$  with  $a_{ij}^* = b_i$ . Go to 6.
9. If the  $j^{\text{th}}$  column in  $A^*$  does not contain any E-type coefficient, go to 6. Otherwise take the smallest number  $r$  of the row with E-type coefficient in the  $j^{\text{th}}$  column of  $A^*$ . Put  $X_{low}(j) = b_r$ ,  $IND(i) = IND(i) + 1$  and  $h_{ij} = b_i$  for  $i = r$  and for each  $i > r$  which  $a_{ij}^* = b_i$ . Go to 6.
10. If  $IND(i) = 0$  for some  $i = 1, \dots, m$  then the system is inconsistent.  $IND(i) = 0$  means that the  $i^{\text{th}}$  equation is in contradiction with the others. Go to 13.
11. If  $IND(i) \neq 1$  for some  $i = 1, \dots, m$ , go to 12. Otherwise the system is consistent with a unique maximal interval solution stretched on  $X_{low}$  and  $X_{gr}$ . Go to 13.
12. The system is consistent,  $X_{gr}$  contains the greatest point solution. A method for computing the lower point solutions and the maximal interval solutions is given in [4].
13. End.

**Theorem 2.** Let the system  $A \cdot X \geq B$  be given.

- i) If the  $j^{\text{th}}$  column of  $A^*$  contains  $a_{kj}^* = G$  then:
  - a)  $X_j = [b_k, 1]$  is a feasible interval for the  $j^{\text{th}}$  component;
  - b)  $x_j \in [b_k, 1]$  implies  $a_{ij} \wedge x_j \geq b_i$  for  $i = k$ , for each  $i > k$  with  $a_{ij} > b_i = b_k$ ;
  - c) for  $i > k$ ; if  $a_{ij} = b_i$  then  $x_j \in [b_k, 1]$  means  $a_{ij} \wedge x_j = b_i$ ;
  - d) for  $i < k$ : if  $a_{ij} = b_i$  then  $x_j \in [b_r, 1]$  means  $a_{ij} \wedge x_j = b_i$ ;
- ii) If the  $j^{\text{th}}$  column of  $A^*$  does not contain any G-type coefficient, but it contains E-type coefficient  $a_{rj}^* = b_r$ , then:
  - a) the feasible interval is  $X_j = [b_r, 1]$ ;
  - b)  $x_j \in [b_r, 1]$  implies  $a_{ij} \wedge x_j = b_i$  for each  $i \leq r$  with  $a_{ij} = b_i$  and there does not exist  $x_j \in L$  such that  $a_{ij} \wedge x_j > b_i$ .
- iii) If the  $j^{\text{th}}$  column in  $A^*$  contains only S-type coefficients, then  $a_{ij} \wedge x_j < b_i$  for each  $x_j \in L$ . □

Let the system  $A \cdot X \geq B$  be given. It is inconsistent iff there exists at least one inequality with neither G- nor E-type coefficient in it. The time complexity function for establishing the compatibility of the system is  $O(m \cdot m)$ . If the system

$A \cdot X \geq B$  is consistent, then it has a unique greatest solution  $X_{gr} = (1, \dots, 1)^t$  and a finite number of lower solutions.

**Algorithm 2** (for solving the system  $A \cdot X \geq B$ ).

1. Enter the matrices  $A_{m \times n}$ ,  $B_{m \times 1}$  ( $|I| = m$ ,  $|J| = n$ ).
2. Computer the matrix  $(A^*B)$ .
3. Erase the marker vector  $IND$  and the help-matrix  $H = (h_{ij})_{m \times n}$ .
4.  $H = A^*$ . For each  $i = 1, \dots, m$ ,  $IND(i)$  is equal to the number of  $h_{ij} \geq b_i$  in the  $i^{th}$  row.
5. If  $IND(i) = 0$  for some  $i \geq m$ , then the system is inconsistent. Go to step 7.
6. Consult [4] to compute  $\{X_{low}\}$ ,  $X_{gr}$  and  $\{X_{max}\}$ .
7. End.

The systems  $A \cdot X \leq B$  and  $A \cdot X < B$  are always consistent: the zero solution  $X_{low} = (0, \dots, 0)^t$  is its unique lower solution.

**Theorem 3.** Let the system  $A \cdot X \leq B$  be given.

- i) If the  $j^{th}$  column of  $A^*$  contains  $a_{kj}^* = G$ , then for any  $x_j \in [0, b_k]$  and:
  - for  $i = k$  the inequality  $a_{ij} \wedge x_j \leq b_i$  holds;
  - for  $i < k$  and  $a_{ij} > b_i$  the inequality  $a_{ij} \wedge x_j \leq b_i$  holds;
  - for any  $a_{ij}^* = b_i$ ,  $a_{ij} \wedge x_j \leq b_i$  for any  $i \in I$ .
- ii) If the  $j^{th}$  column in  $A^*$  does not contain any  $G$ -type coefficient but it contains  $E$ -type coefficient then  $L$  is a feasible interval for the  $j^{th}$  component of the solution and  $x_j \in L$  means  $a_{ij} \wedge x_j \leq b_i$  for each  $i = 1, \dots, m$ .
- iii) If the  $j^{th}$  column in  $A^*$  contains only  $S$ -type coefficients then  $a_{ij} \wedge x_j < b_i$  for each  $i = 1, \dots, m$  and any  $x_j \in X_j = L$ . □

For solving  $A \cdot X > B$  and  $A \cdot X < B$  a slight modification of these results is valid [4].

**Theorem 4.** The following problems are algorithmically decidable in polynomial time for the system  $A \cdot X \perp B$ :

- i) whether the system is consistent or not;
- ii) if the system is consistent, computing all its solutions.
- iii) obtaining the numbers of the contradictory equations if the system is inconsistent. □

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