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ON PROPERTIES OF OBJECTS AND FUZZY SETS

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Abstract: This paper is based on an approach to formulating a fuzzy sets theory basing on a model of fuzzy set suggested in Orlovski, 1990 [2]. That formulation gives transparent interpretation of membership degrees and in this formulation a theory of fuzzy sets is not treated as a set theory.

Keywords: Membership function, composition of properties

1. INTRODUCTION

To apply methods of the fuzzy sets theory to analyze an informational situation, one should "encode" his subjective knowledge in terms of membership degrees. And this is a difficult task if one does not have a clear interpretation of these degrees.

As G. Shafer, 1981 [3] put in the context of probabilistic models, such interpretation should be based on descriptions of types of "canonical" informational schemes for which the membership function appears to be a natural mathematical description. Basing on such interpretation, an expert can compare an appropriate canonical scheme with the type of information to be analyzed, i.e. perform a "thought experiment", and accept an appropriate membership function.

Refering to the possibility theory, R. Giles, 1983 [1] wrote: "...This does not mean that an objective procedure for determining such a possibility must be provided – indeed, most fuzzy concepts are a matter of opinion – but a procedure should be laid down that allows an agent to translate his beliefs into a numerical possibility, and gives some assurance that two agents who assign the same possibility value do really have beliefs that agree in some tangible way."

The theory of fuzzy sets in its present form is commonly understood as a set theory, where set operations such as intersection, union, etc. play the basic role. The difficulty with such an approach lies in that once introduced the set operations must invariably be used within the theory; but practice shows that it is frequently impossible to fulfill this requirement in applied problems. On the other hand, the ease with which different definitions of the same operations are frequently mixed up within one line of analysis is not logically justified within such set-theoretic framework.

In this paper basing on a different approach we repeat some basic definitions from Orlovski, 1990 [2], then apply the concepts introduced to describe independence of properties (and fuzzy sets), and then give some informal considerations concerning the structure of spaces of elementary properties.

2. SOME BASIC DEFINITIONS

Let P be a set and B be a class of subsets of P such that:

(a) $\Pi \in B \Rightarrow P \setminus \Pi \in B$;

(b) $\emptyset \in B$.

We refer to B as a complete class of subsets of set P. Function $\mu : B \to [0,1]$ is called a pseudomeasure on (P, B) iff it satisfies the following conditions:

(1) $\mu(\emptyset) = 0$;

(2) $\Pi_1 \subseteq \Pi_2 \Rightarrow \mu(\Pi_1) \leq \mu(\Pi_2)$;

(3) $\mu(P) = 1$.

With each pseudomeasure μ on (P, B) we associate a function $\mu^* : B \to [0,1]$ as follows: $\mu^*(\Pi) = 1 - \mu(P \setminus \Pi)$, $\Pi \in B$. Clearly, this function is also a pseudomeasure and we refer to μ^* as the pseudomeasure dual to μ. As can easily be seen, μ and μ^* are mutually dual. We call triplet (P, B, μ) space with a pseudomeasure.

Let (P_i, B_i, μ_i), $i = 1,2$ be two spaces with pseudomeasures. Consider direct product $P = P_1 \times P_2$ and denote by B^1 class of (rectangular) subsets of set P such that

$$\Pi \in B^1 \Rightarrow \Pi = \Pi_1 \times \Pi_2, \quad \Pi_i \in B, \quad i = 1,2.$$

We add to class B^1 sets of the form $P \setminus \Pi$, $\Pi \in B^1$ and obtain a complete class of subsets of set P, that we shall refer to as the complete class of subsets induced by B^1.

The next step is to equip (P, B) with a pair of mutually dual pseudomeasures, based on μ_1 and μ_1. For any set $\Pi \in B^1$, i.e. $\Pi = \Pi_1 \times \Pi_2$, $\Pi_i \in B_i$, $i = 1,2$, we put:

$$\mu(\Pi) = \min\{\mu_1(\Pi_1), \mu_2(\Pi_2)\}, \quad \mu^*(\Pi) = \min\{\mu_1^*(\Pi_1), \mu_2^*(\Pi_2)\}.$$

Next we extend this definition to the complete class B that includes also sets of the form $P \setminus \Pi$, $\Pi \in B^1$, to make these functions mutually dual:

$$\begin{aligned}\mu(P\setminus\Pi) &= 1-\mu^*(\Pi) = 1-\min\{\mu_1^*(\Pi_1),\mu_2^*(\Pi_2)\} = \\ &= \max\{\mu_1(P_1\setminus\Pi_1),\mu_2(P_2\setminus\Pi_2)\},\\ \mu^*(P\setminus\Pi) &= 1-\mu(\Pi) = 1-\min\{\mu_1(\Pi_1),\mu_2(\Pi_2)\} = \\ &= \max\{\mu_1^*(P_1\setminus\Pi_1),\mu_2^*(P_2\setminus\Pi_2)\}.\end{aligned}$$

We call μ and μ^* product pseudomeasures, and space (P,B,μ) - product of spaces (P_i,B_i,μ_i), $i=1,2$.

The above definition of product pseudomeasures has an intuitively clear justification that we give later on.

Suppose now, that we have a set X of objects and a space with pseudomeasure (P,B,μ). With respect to X we call elements of P *elementary properties*, and elements of B - *collections of elementary properties.*

A set valued mapping a from space of elementary properties to class 2^X of all subsets of X such that for any $x\in X$ set $a^{-1}(x)=\{p|x\in a(p)\}$ belongs to B, is called decomposable property defined for X on (P,B,μ). Equivalently, we say that a fuzzy subset a is defined for set of objects X, this fuzzy set being a collection of objects form X showing decomposable property a. We shall understand proposition $x\in a$ as any of the following equivalent assertions: "x shows decomposable property a", and "x belongs to fuzzy set a".

A function $\nu_a : X\to[0,1]$ defined as $\nu_a(x)=\mu\{a^{-1}(x)\}$ is called the membership function of a. A value $\nu_a(x)$ of this function is interpreted as the degree to which proposition $x\in a$ is true, i.e. the degree to which object x shows decomposable property a, or, equivalently, the degree to which object x belongs to fuzzy subset a.

3. DECOMPOSABLE PORPERTIES ON PRODUCTS OF SPACES OF ELEMENTARY PROPERTIES

Let x be a set of objects, (P_a,B_a), (P_b,B_b) be spaces of elementary properties, and $a:P_a\to 2^X$, $b:P_b\to 2^X$ be properties-objects mappings. Denote by P direct product $P_a\times P_b$ and by B - the complete class of subsets of P, corresponding to B_a, B_b. Suppose a mapping $c:P\to 2^X$ and a pseudomeasure $\mu_{ab}:B\to[0,1]$ (together with the respective dual pseudomeasure μ^*_{ab}) are defined on B. Then we can say that a decomposable property c is defined for set of objects X such that its set of elementary properties is the direct product $P_a\times P_b$. The corresponding membership function of c has the form:

$$\nu_c(x)=\mu_{ab}(c^{-1}(x)),\quad x\in X.$$

Suppose now that the decomposable property c in question is logically expressed as $a \cap b$, i.e. has the meaning of having properties a and b simultaneously. This means that the mapping c has the form: $c(p_a, p_b) = a(p_a) \cap b(p_b)$, and the pseudomeasure μ_{ab} is such that $\mu_{ab}(a^{-1}(x) \times b^{-1}(x))$ is the degree to which x shows properties a and b simultaneously.

Basing on μ_{ab} and μ^*_{ab}, let us introcude the following functions on B_a and B_b respectively:

$$\mu_a(\Pi_a) = \mu_{ab}(\Pi_a x P_b); \qquad \mu_b(\Pi_b) = \mu_{ab}(P_a x \Pi_b),$$
$$\mu^*_a(\Pi_a) = \mu^*_{ab}(\Pi_a x P_b); \qquad \mu^*_b(\Pi_b) = \mu^*_{ab}(P_a x \Pi_b).$$

These functions can be called projections of pseudomeasures μ_{ab} and μ^*_{ab} on B_a and B_b, respectively. As can easily be shown, (μ_a, μ^*_a), (μ_b, μ^*_b) are pairs of mutually dual pseudomeasures on B_a and B_b, respectively.

The pseudomeasures introduced above possess the following properties for any $\Pi_a \in B_a$, $\Pi_b \in B_b$:

(1) $\min\{\mu_a(\Pi_a),\ \mu_b(\Pi_b)\} \geq \mu_{ab}(\Pi_a \times \Pi_b)$;

(2) $\max\{\mu_a(P_a \setminus \Pi_a),\ \mu_b(P_b \setminus \Pi_b)\} \geq \mu_{ab}(P \setminus (\Pi_a \times \Pi_b))$.

Similar inequalities (1′) and (2′) are valid for μ^*.

Consider now decomposable properties a and b defined for set X by the above mappings $a(\cdot)$ and $b(\cdot)$ and pseudomeasures μ_a, μ_b. We shall say that properties a and b are *independent* on set X iff for any $x \in X$:

$$\mu_{ab}(a^{-1}(x) \times b^{-1}(x)) = \min\{\mu_a(a^{-1}(x)),\ \mu_b(b^{-1}(x))\},$$
$$\mu^*_{ab}(a^{-1}(x) \times b^{-1}(x)) = \min\{\mu^*_a(a^{-1}(x)),\ \mu_b(b^{-1}(x))\}.$$

As can easily be shown, the independence of properties a and b implies that we have equalities also in similar inequalities (1′) and (2′). Note, that if decomposable properties a and b are independent on set X, then they are independent on any subset of X.

Suppose we have two decomposable properties a and b defined for set X of objects, and we would like to consider a decomposable property c for X consisting in showing both properties a and b simultaneously. Clearly, a mapping $c : P_a \times P_b \to X$ corresponding to such property should have the form $c(p_a, p_b) = a(p_a) \cap b(p_b)$. To complete the definition of this property we should define a pair of dual pseudomeasures μ_c and μ^*_c on $P_a \times P_b$. Clearly, these pseudomeasures should be such that their projections on P_a and P_b coincide with the respective pseudomeasures for properties a and b. In other words, we should have:

$$\sup_{\Pi_a \in B_a} \mu_c(\Pi_a \times \Pi_b) = \mu_a(\Pi_a), \qquad \sup_{\Pi_a \in B_a} \mu^*_c(\Pi_a \times \Pi_b) = \mu^*_a(\Pi_a),$$
$$\sup_{\Pi_b \in B_b} \mu_c(\Pi_a \times \Pi_b) = \mu_b(\Pi_b), \qquad \sup_{\Pi_b \in B_b} \mu^*_c(\Pi_a \times \Pi_b) = \mu^*_b(\Pi_b).$$

Otherwise, pseudomeasures μ_c, μ_c^* can be chosen arbitrarily basing on the meaning that we put into property c. Possible examples for pseudomeasure μ_c can be:

$$a \text{ and } b \text{ independent}: \mu_c(\Pi_a \times \Pi_b) = \min\{\mu_a(\Pi_a), \mu_b(\Pi_b)\},$$

$$\mu_c(\Pi_a \times \Pi_b) = \mu_a(\Pi_a) \cdot \mu_b(\Pi_b),$$

$$\mu_c(\Pi_a \times \Pi_b) = \max\{\mu_a(\Pi_a) + \mu_b(\Pi_b) - 1, 0\}, \text{ etc.}$$

As can easily be seen, the class of such possible pseudomeasures includes (but not limited to) those obtained from $\mu_a(\Pi_a)$, $\mu_b(\Pi_b)$ using any of so called T-norms.

4. INFORMAL CONSIDERATIONS ABOUT THE STRUCTURES OF SETS OF ELEMENTARY PROPERTIES

Suppose we have set X of objects, and two sets P, Q of elementary properties. Suppose also that we have two object-properties mappings: $a : P \to 2^X$ and $b : Q \to 2^X$, such that $a(p)$, $b(q)$ are subsets of objects having properties p and q respectively. Consider direct product $P \times Q$ and a mapping $c : PxQ \to 2^X$ such that $c(p,q)$ is the subset of objects each having properties p and q simultaneously.

For any object $x \in X$ subset $c^{-1}(x) \subseteq P \times Q$ is the collection of elementary properties that x has. Denote by c_a^{-1}, $c_b^{-1}(x)$ projections of $c^{-1}(x)$ into P and Q, respectively. It can easily be shown that for any $x \in X$ set c^{-1} is a rectangular subset of $P \times Q$, i.e. $c^{-1}(x) = c_a^{-1}(x) \times c_b^{-1}(x)$. Rather than formally proving this, we illustrate this fact using the following example.

Suppose the subset $C = P \times Q$ contains pairs of properties:

$$C \supseteq \{(p_1, q_1), (p_2, q_2), (p_3, q_3), (p_1, q_4)\}.$$

This means, in particular, that object x has properties p_1 and q_3, i.e. the pair (p_1, q_3) also belongs to C. Continuing this, we can easily see that C is indeed a direct product of its projections into P and Q, respectively. In other words, rectangular hall of any subset of C is also subset of C. Having this in mind, one can say that set C of all pairs of properties of an object is, in fact, fully defined by any its subset that has the same projections on P and Q as C.

Now we give some intuitive grounds for defining product pseudomeasure using operation min. Let us call pairs of properties (p_1, q_1), $(p_2, q_2) \in P \times Q$ independent iff $p_1 \neq p_2$ and $q_1 \neq q_2$. Suppose first that P and Q each contain M properties. Then, as can easily be seen, any inclusion-maximum subset of independent properties in $P \times Q$ contains M elements and fully describes the whole set $P \times Q$ as has been noted in the above paragraph. Take a rectangular subset $\Pi = A \times B$ of set $P \times Q$ with A containing m elements, and B containing n elements. Then the total number of independent pairs of properties in Π is equal to $\min\{m, n\}$. Assume $\mu(A) = m/M$, and $\mu(B) = n/M$. If we understand pseudomeasure of Π as the relative number of independent properties contained in it, then we have:

$$\mu(\Pi) = \frac{\min\{m, n\}}{M} = \min\{\frac{m}{M}, \frac{n}{M}\} = \min\{\mu(A), \mu(B)\}$$

Suppose now that P contains M elements, and Q contains N elements. Let K be the minimal number such that $K = \lambda_M \cdot M = \lambda_N \cdot N$ with λ_M, λ_N being integers. Using them we scale importances of elementary properties to comparable units by "spliting" every elementary property in P into λ_M elements, and every elementary property in Q - into λ_N elements. As a result we have that both P and Q contain K new elements each. Then with the above understanding of pseudomeasure of Π as relative number of independent properties contained in it, we obtain:

$$\mu(\Pi) = \frac{\min\{\lambda_M \cdot m, \lambda_N \cdot n\}}{\min\{\lambda_M \cdot M, \lambda_N \cdot N\}} = \frac{\min\{\lambda_M \cdot m, \lambda_N \cdot n\}}{K} =$$
$$= \min\{m/M, n/N\} = \min\{\mu(A), \mu(B)\}.$$

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