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THE UNIFYING ROLE OF FUZZY LOGIC IN FUZZY SET THEORY

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Abstract: In this paper, we sum up our opinion on the problem of unification of fuzzy set theory. We are convinced that such a unification is desirable and possible and gives reasons for the use of first–order fuzzy logic which is a nontrivial generalization of classical logic inheriting many of its important properties including the syntactico–semantical completeness and, moreover, it is the only one system of this kind, up to isomorphism.

Keywords: Fuzzy logic, fuzzy set theory, approximate reasoning, operations on fuzzy sets, t-norms.

1. WHY UNIFICATION

Fuzzy set theory is now a quite popular theory which has found many interesting applications, e.g. controlling of washing machine, underground train, recognition of handwritten letters, and many others. Yet, it is not a unique theory with clear axioms, operations and terms. Many authors devote their work to analysis of various algebraic structures of membership degrees, various kinds and sorts of operations with fuzzy sets, especially conjunction, implication and negation (complement), etc. These researches are often made on the basis of intuition or several assumed axioms. The result is usually compared with the results obtained on the basis of other (different) axioms. However, no final conclusion can be made since there are no reasonable criteria for the decision, which structure is best. The problem consists in the fact that the interval $\langle 0, 1\rangle$ which usually serves us as a basis for membership degrees is extremely rich. It is possible to define uncountably many operations on it. Such a situation is rather unpleasant: if everything is possible and good from some point of view then it is not possible to develop fuzzy set theory as a concise theory with its own style of reasoning and clear subject of research. Consequently, such a blurred theory can hardly be interesting enough and probably will not be widely accepted[1]. We conclude that unification of fuzzy set theory resulting in a concise, transparent and non-trivial theory with clear subject of interest is necessary.

In this paper, we propose the way which might lead to such a desired unification. Let us stress that an attempt to present fuzzy set theory as a unified theory in this sense has already been done in [5].

Let us first say a few words about the subject of fuzzy set theory. When regarding the world[2], a man encounters various phenomena. The phenomena which can be viewed as themselves having their own "personality" differentiating them from the surrounding world are called *objects*. Objects are accompanied by other phenomena which are called *properties*. The properties can be characterised by groupings of elements being accompanied by them. In other words, given a property φ of objects, there is a grouping

$$X = \{x; \varphi(x)\} \tag{1}$$

containing all the elements x which have the property φ. However, except for few special cases, the properties φ represent vague phenomena. This means that the corresponding groupings X from (1) are not sets — their elements cannot be written down on a list and in general, it is not possible to decide unambiguously whether $\varphi(x)$ for every element x, or not[3]. For example, consider $\varphi :=$ *red*. Then it is by no means possible to decide about any wave length whether it represents red colour or not.

The situation can be made more transparent if we consider a scale L of truth values. Given an object x, we may assign a degree of truth $a \in L$ to any $\varphi(x)$ and say that $\varphi(x)$ is *true* in the degree a. Thus, we describe the unsharp, blurred grouping X by a certain function A with the range L. The domain of A is a *universe* of objects which can be taken into account and it is resonable to consider it to be a set. We may conclude that *fuzzy set theory is a theory whose subject of study are vague phenomena. They are studied using certain, quite specific functions.*

Many studies have been devoted to the structure of truth values. Good reasons which have been discussed in [7, 5, 4, 3] lead us to the assumption that truth values should form a residuated lattice

$$\mathcal{L} = \langle L, \vee, \wedge, \otimes, \rightarrow, 0, 1\rangle. \tag{2}$$

[1] Taking all in all, the latter is true especially among mathematicians working in other areas of mathematics.

[2] By "regarding" we mean any way of learning the world, i.e. not only regarding it by senses.

[3] $\varphi(x)$ should be read as "the element x has the property φ".

(for the properties and exact definition of residuated lattice see the above cited literature). Let us stress that $\mathcal{L}$ preserves most of the properties demanded on the structure of membership degrees from various points of view. In the case of $L = \langle 0, 1\rangle$, the operations $\otimes$ (bold product) and $\rightarrow$ (residuation) are defined as follows

$$\begin{aligned} a \otimes b &= 0 \vee (a + b - 1) \\ a \rightarrow b &= 1 \wedge (1 - a + b) \end{aligned} \tag{3}$$

for all the $a, b \in \langle 0, 1\rangle$.

Theorem 1. *Let $\mathcal{L}'$ be a residuated lattice with $L' = \langle 0, 1\rangle$ and the operation $\rightarrow'$ be continuous. Then $\mathcal{L}'$ is isomorphic with with the residuated lattice $\mathcal{L}$ having $L = \langle 0, 1\rangle$ and endowed with the operations (3).* □

This theorem is very important since it drastically reduces the choice of operations to be considered as basic in fuzzy set theory.

It may be clear from the above discussion that fuzzy set theory is closely connected with many–valued (fuzzy) logic. This is quite natural since classical set theory is closely connected with classical logic, as well. The nature of this connection in both theories is the same.

Let us now say some words about fuzzy logic.

2. SOME COMMENTS ON FIRST–ORDER FUZZY LOGIC

There are several systems of many–valued (fuzzy logic) which have various common points and properties. Among them, the system presented in [7] and extended to first–order one in [4] is of special interest for us. Its main properties are the following:

- it is a nontrivial generalization of classical logic,
- it preserves as many properties of classical logic as possible,
- it meets most of the intuitive requirements,
- it is sufficiently rich and interesting.

From now, the lattice $\mathcal{L}$ is assumed to be a chain.

The most outstanding feature of this system of fuzzy logic is its syntactico–semantical completeness, i.e. the generalisation of Gödel's completeness theorem holds true:

Theorem 2.

$$T \vdash_a A \quad iff \quad T \models_a A$$

holds true for every theory T and a formula A. □

In words — a formula A is true in the degree a iff it is a theorem (provable) in the degree a.

There are also other theorems known from classical logic which are suitably generalized in first order fuzzy logic, namely *deduction theorem, closure theorem, extension of T by a new predicate*, etc. The following general theorem is very important (see [7]).

Theorem 3. *Let $\mathcal{L}$ be a chain. Then it is not possible to find a fuzzy logic with complete syntax provided that any of the following properties is true:*

(a) *The chain $\mathcal{L}$ is infinitely countable.*

(b) *The chain $\mathcal{L}$ is uncountable and the residuation operation $\rightarrow$ is not continuous.* □

Thus, our choice is limited. The reason for our choice of $\mathcal{L}$ with the operations (3) follows from Theorems 1 and 3. So far, it is not known whether Theorem 2 holds in the case that $\mathcal{L}$ is not a chain. Nevertheless, the case of $L = \langle 0, 1\rangle$ is the most important and natural, and it is used in all the applications of fuzzy set theory known so far. Let us remark that first-order fuzzy logic is isomorphic with classical logic if $L = \{0, 1\}$.

There is one more important feature of first–order fuzzy logic which makes it much more tractable. There are four basic connectives — *disjunction* $\vee$, *conjunction* $\wedge$, *bold conjunction* & and *implication* $\Rightarrow$ — interpreted using the four basic operations in the residuated lattice. Negation is a derived connective defined by $\neg A := A \Rightarrow \mathbf{0}$ ($\mathbf{0}$ is a symbol for the truth value 0).

However, it is possible to enrich $\mathcal{L}$ by new additional n-ary operations

$$o : L^m \longrightarrow L$$

provided that they fulfil the *fitting condition:* there are $p_1, \ldots, p_n$ such that

$$(a_1 \leftrightarrow b_1)^{p_1} \otimes \ldots \otimes (a_m \leftrightarrow b_m)^{p_m} \leq o(a_1, \ldots, a_m) \leftrightarrow o(b_1, \ldots, b_m) \qquad (4)$$

holds for every $a_i, b_i \in L, i = 1, \ldots, m$ where

$$a \leftrightarrow b = (a \rightarrow b) \wedge (b \rightarrow a)$$

(biresiduation) and the power is taken with respect to the operation $\otimes$.

The operations o then serve us as the interpretations of new additional n-ary connectives of first–order fuzzy logic. At the same time, fitting condition assures us that completeness theorem is not harmed.

Thus, we may naturally add connectives serving us, e.g. as interpretations of linguistic modifiers (*very, slightly,* etc.) and, together with the quantifiers $\forall$ and $\exists$, also as interpretations of generalized linguistic quantifiers (*most, many,* etc.).

Note that all the operations used in first–order fuzzy logic are fitting, i.e. they fulfil the condition (4). This is another significant restriction layed on the considered operations.

3. THE IMPACT OF FUZZY LOGIC ON FUZZY SET THEORY AND ITS APPLICATIONS

We may now sum up all the arguments for the use of first–order fuzzy logic as a frame for fuzzy set theory:

1. This logic is syntactico–semantically complete. Thus, it may become a sound language of fuzzy set theory, analogously as classical logic is a language of classical set theory. Many important and useful properties of classical logic are preserved. On the other side, many proofs of the theorems are nontrivial and are not immediately seen.
2. If $L = \langle 0, 1 \rangle$ then, up to isomorphism, it is the only system with such properties.
3. The residuated lattice of truth values enriched by additional n-ary operations
$$\mathcal{L}' = \langle L, \vee, \wedge, \otimes, \rightarrow, \{o_i; i \in I\}, \bigvee, \bigwedge, 0, 1 \rangle.$$
includes a great deal of the structures considered so far as potential structures of membership degrees (or truth values).
4. The additional operations include most of the continuous t-norms. The rejection of the rest of t-norms is desirable since we need a suitable restriction (cf. discussion in the first section and also in [6].
5. Using first–order fuzzy logic, we obtain a unified and transaprent view on the operations on fuzzy sets.
6. One of the most outstanding applications of fuzzy set theory is approximate reasoning (realised mainly in the form of fuzzy controller). First–order fuzzy logic naturaly justifies its methods and gives us suggestions for further research.
7. This logic may become also an origin for further reasoning in other respects, for example in instrumental characterisation of natural infinity (see [2]).

Let us briefly illustrate the items 4. and 5. Let $A, B \underset{\sim}{\subset} U$. The basic operations with fuzzy sets are:

union

$$C = A \cup B \quad \text{iff} \quad Cx = Ax \vee Bx$$

intersection

$$C = A \cap B \quad \text{iff} \quad Cx = Ax \wedge Bx$$

bold intersection

$$C = A \otimes B \quad \text{iff} \quad Cx = Ax \otimes Bx$$

residuum

$$C = A \ominus B \quad \text{iff} \quad Cx = Ax \rightarrow Bx.$$

complement[4]

$$\overline{A} = A \ominus \emptyset.$$

For any kind of an m-ary operation $o : L^m \longrightarrow L$ we may give the following general definition: The operation o is a *basis* of the operation O assigning a fuzzy set $C \underset{\sim}{\subseteq} U$ to $A_1, \ldots, A_m$ if

$$C = O(A_1, \ldots, A_m) \quad \text{iff} \quad Cx = o(A_1 x_1, \ldots, A_m x_m)$$

holds for every $x \in U$.

Theorem 4. *Let $\{o_d; d \in I\}$ be a set of operations enriching the lattice $\mathcal{L}$ which are the bases of the corresponding operations $\{O_d; d \in I\}$. Then*

$$\langle \mathcal{F}(U), \cup, \cap, \otimes, \ominus, \emptyset, U, \{O_d; d \in I\} \rangle$$

is a residuated lattice enriched by the operations $\{O_d; d \in I\}$. □

Examples of additional operations on fuzzy sets:

Algebraic product

$$C = A \cdot B \quad \text{iff} \quad Cx = Ax \cdot Bx.$$

Algebraic sum

$$C = A \oplus B \quad \text{iff} \quad Cx = Ax + Bx - Ax \cdot Bx.$$

Bounded difference

$$C = A \ominus B \quad \text{iff} \quad Cx = 0 \vee (Ax - Bx),$$

The *approximate reasoning* deals with fuzzy sets of closed formulae

$$\{a_t / A_x[t]; t \in M_J\} \tag{5}$$

where $A(x)$ is a given open formula. *Deduction in approximate reasoning* is then a deduction in the fuzzy theory given by fuzzy sets (5) of special axioms.

[4] If $L = \langle 0,1 \rangle$ then this definition gives $\overline{A}x = 1 - Ax$ for all $x \in U$.

For example, the *generalized modus ponens* is a partial derivation of the fuzzy set of formulae

$$\{b_s/B_y[s]; s \in M_J\} \qquad M_J \text{ is a set of all terms}$$

from

$$\{a_t/A_x[t]; t \in M_J\}$$
$$\{c_{ts}/(A_x[t] \Rightarrow B_y[s]); t, s \in M_J\}$$

using a set $\{w_{ts}; t, s \in M_J\}$ of proofs[5]

$$w_{ts} := A_x[t]\ [a_t; SA], A_x[t] \Rightarrow B_y[s]\ [c_{ts}; SA], B_y[s]\ [a_t \otimes c_{ts}; r_{MP}]$$

Then

$$b_s = \bigvee\{a_t \otimes c_{ts}; t \in M_J\},$$

$s \in M_J$.

4. CONCLUSION

In this paper, we sum up our opinion on the problem of unification of fuzzy set theory. We are convinced that such a unification is desirable and possible and give reasons for the use of first-order fuzzy logic presented in [4] as a suitable language which may lead to the desired unification. The unification consists mainly in proper restrictions made on the choice of operations on fuzzy sets and the use of other notions and theorems of fuzzy logic (rules of inference, proof, etc.). This is important especially in logical applications, e.g. in expert systems, approximate reasoning, fuzzy-PROLOG, etc.

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