t-NORM–BASED OPERATIONS ON FUZZY SETS*

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Abstract: The goal of this presentation is to review certain results concerning the t-norm–based operations of fuzzy sets. We present a generalization of Nguyen's results regarding the level sets of two–place functions defined via sup–t-norm convolution, and also give an exact calculation formula for extended addition of fuzzy intervals of LR–type.

Keywords: Extension principle, triangular–norm, level set, LR fuzzy interval.

1. INTRODUCTION

Solving practical problems one has to decide which of the t-norms to calculate with. To different problems may fit different t-norms. In the majority of cases they use the "min"–norm ($T(x,y) = \min\{x,y\}$) introduced by L. Zadeh which is quite natural and the most simple to handle. But the "min"–norm is the greatest one in the sense that $T(x,y) \leq \min\{x,y\}$ for all the t-norms T. This property of "min"–norm may cause too fast growing of uncertainty in calculations. A very important feature of the approach by t-norms is that it provides means of controlling the growth of uncertainty and prevents variables from simultaneous shift off their most significant values. In this respect, the various formulas of t-norm–based operations yield practical tools for achieving this control and are very meaningful.

Let $X \neq \emptyset$, $Y \neq \emptyset$ and $Z \neq \emptyset$ be three universes and the mapping $* : X \times Y \to Z$ an operation between X and Y taking its value in Z. The arithmetical operations are $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$ mappings, where $\mathbb{R}$ denotes the real line. Denote by $\mathcal{F}(X)$, $\mathcal{F}(Y)$ and $\mathcal{F}(Z)$ the set of all fuzzy subsets of X, Y and Z, respectively and let $A \in \mathcal{F}(X)$,

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$B \in \mathcal{F}(Y)$. We can fuzzify the operation $*$ defining via Zadeh's extension principle an $\mathcal{F}(X) \times \mathcal{F}(Y) \to \mathcal{F}(Z)$ operation as follows

$$\mu_{A*B}(z) = \sup_{x*y=z} T(\mu_A(x), \mu_B(y)) \tag{1.1}$$

where $A * B \in \mathcal{F}(Z)$; μ_A, μ_B, μ_{A*B} are the membership functions of fuzzy sets A, B and $A * B$, respectively and $T : [0,1] \times [0,1] \to [0,1]$ is an arbitrary t-norm.

With respect to the possibility theory, membership functions μ_A and μ_B are considered as possibility distributions of some variables u and v taking their values in X and Y, where $\mu_A(x)$ and $\mu_B(y)$ correspond to the grades of possibility of choosing x and y as suitable values for u and v respectively. In this sense, using in (1.1) $T = \min$ we get an operation on two noninteractive possibility distributions. However, using a t-norm in general we get an operation on two weakly noninteractive possibility ditsributions [1].

2. USING LEVEL SETS OF FUZZY SETS

A natural way of practical computations on fuzzy sets is to use α-cuts (or α-level sets). Recall that the α-cut of the fuzzy set $A \in \mathcal{F}(X)$ is

$$[A]^\alpha = \{x \in X \mid \mu_A(x) \geq \alpha\} \qquad \alpha \in]0,1].$$

Let X and Y be topological spaces and denote by $\mathcal{F}(X, \mathcal{K})$, $\mathcal{F}(Y, \mathcal{K})$ the set of fuzzy subsets of X and Y, respectively, having compact support (we mean the closed support of a fuzzy set: $Supp(A) := \overline{\{x \in X \mid \mu_A(x) > 0\}}$) and upper semicontinuous (u.s.c. for short) membership function.

Nguyen [5] investigated the operations on noninteractive fuzzy sets from the point of view of α-cuts. He gave a necessary and sufficient condition for obtaining the equality

$$[A * B]^\alpha = [A]^\alpha * [B]^\alpha \qquad \alpha \in]0,1]$$

where $[A]^\alpha * [B]^\alpha = \{z = x * y \mid (x,y) \in [A]^\alpha \times [B]^\alpha\}$.

Generalizing this result to the case of weakly noninteractive fuzzy numbers [3] a necessary and sufficient condition can also be given for obtaining the corresponding equality:

$$[A * B]^\alpha = \bigcup_{T(\xi,\eta) \geq \alpha} [A]^\xi * [B]^\eta \qquad \alpha \in]0,1] \tag{2.1}$$

Theorem 2.1.. *A necessery and sufficient condition for obtaining the equality (2.1) is that* $\sup_{x*y=z} T(A(x), B(y))$ *is attained for all* $z \in Z$. □

Proof. (i) Necessity. Let $z \in Z$ and

$$(A * B)(z) = \sup_{x*y=z} T(A(x), B(y)) = t$$

Then,

$$z \in [A * B]^t = \bigcup_{T(\xi,\eta)\geq t} [A]^\xi * [B]^\eta$$

by hypothesis. Therefore, there exist ξ_0, η_0 such that $T(\xi_0, \eta_0) \geq t$ and $z \in [A]^{\xi_0} * [B]^{\eta_0}$ i. e. there exists $(x_0, y_0) \in [A]^{\xi_0} \times [B]^{\eta_0}$ such that $x_0 * y_0 = z$. But

$$t = \sup_{x*y=z} T(A(x), B(y)) \geq T(A(x_0), B(y_0)) \geq T(\xi_0, \eta_0) \geq t$$

and thus $T(A(x_0), B(y_0)) = t$.
(ii) Sufficiency. Let

$$z \in \bigcup_{T(\xi,\eta)\geq \alpha} [A]^\xi * [B]^\eta$$

that is, there exist ξ_0, η_0 such that $T(\xi_0, \eta_0) \geq \alpha$ and $z \in [A]^{\xi_0} * [B]^{\eta_0}$. However, if $(x_0, y_0) \in [A]^{\xi_0} \times [B]^{\eta_0}$, then

$$(A * B)(z) = \sup_{x*y=z} T(A(x), B(y)) \geq T(A(x_0), B(y_0)) \geq T(\xi_0, \eta_0) \geq \alpha$$

and thus $z \in [A * B]^\alpha$.

On the other hand, let $z \in [A * B]^\alpha$, i.e.

$$\sup_{x*y=z} T(A(x), B(y)) \geq \alpha$$

By hypothesis, there exists $(x_0, y_0) \in X \times Y$ such that $x_0 * y_0 = z$ and

$$T(A(x_0), B(y_0)) = \sup_{x*y=z} T(A(x), B(y)) \geq \alpha$$

thus by taking $\xi_0 := A(x_0)$ and $\eta_0 := B(y_0)$, we have $T(\xi_0, \eta_0) \geq \alpha$, i.e. $(x_0, y_0) \in [A]^{\xi_0} \times [B]^{\eta_0}$ and $z \in [A]^{\xi_0} * [B]^{\eta_0}$, implying that $z \in \bigcup_{T(\xi,\eta)\geq\alpha}[A]^\xi * [B]^\eta$. ■

Now, we show that equality (2.1) holds for all continuous operations and u.s.c. t-norms in the class of fuzzy sets having compact support end u.s.c. membership function.

Theorem 2.2. *If $* : X \times Y \to Z$ is continuous and the t-norm T is upper semicontinuous, then (2.1) holds for all $A \in \mathcal{F}(X, \mathcal{K})$ and $B \in \mathcal{F}(Y, \mathcal{K})$.* □

Proof. By virtue of previous theorem it is sufficient to show, that $\sup_{x*y=z} T(A(x), B(y))$ is attained for all $z \in Z$.

Denote by φ the mapping $(x, y) \mapsto T(A(x), B(y))$. Obviously,

$$\sup_{x*y=z} T(A(x), B(y)) = \sup_{\substack{x*y=z \\ (x,y)\in Supp(A)\times Supp(B)}} \varphi(x, y)$$

since $T(A(x), B(y)) = 0$ outside of the set $Supp(A) \times Supp(B)$.

However, $Supp(A) \times Supp(B)$ is compact and $\{(x, y) \mid x * y = z\}$ is closed by continuity of $*$; hence $\{(x, y) \mid x * y = z\} \cap Supp(A) \times Supp(B)$ is compact too.

T is non-decreasing, u.s.c., A and B are also u.s.c., hence φ is u.s.c. as well. Thus φ assumes its maximum on the compact set

$$\{(x, y) \mid x * y = z\} \cap Supp(A) \times Supp(B)$$

for all $z \in Z$. ∎

3. USING FUZZY INTERVALS OF LR–TYPE

An another way of treating with fuzzy sets in practical computations is to use fuzzy numbers or fuzzy intervals of special type such as LR fuzzy intervals. The addition rule of LR-fuzzy intervals is well-known in the case of "min"-norm [1]. We give in this section exact calculation formulas for t-norm–based addition of special LR-fuzzy intervals. Recall that an LR fuzzy interval $A = (a^-, a^+, \alpha, \beta)_{LR}$ has a membership function

$$A(x) = \begin{cases} 1 & \text{if } x \in [a^-, a^+] \\ L\left(\frac{a^- - x}{\alpha}\right) & \text{if } x \in [a^- - \alpha, a^-] \\ R\left(\frac{x - a^+}{\beta}\right) & \text{if } x \in [a^+, a^+ + \beta] \\ 0 & \text{otherwise} \end{cases} \tag{3.1}$$

where $[a^-, a^+]$ is the peak of A; a^- and a^+ are the lower and upper modal values; L and R : $[0, 1] \to [0, 1]$ are the shape functions with $L(0) = R(0) = 1$ and $L(1) = R(1) = 0$ which are non-increasing, continuous mappings.

A t-norm T is said to be Archimedean iff T is continuous and $T(x, x) < x$, $\forall x \in]0, 1[$. Every Archimedean t-norm T is representable by a continuous and decreasing function $g : [0, 1] \to [0, \infty[$ with $g(1) = 0$ and

$$T(x, y) = g^{[-1]}(g(x) + g(y))$$

where $g^{[-1]}$ is the pseudo-inverse of g, defined by

$$g^{[-1]}(y) = \begin{cases} g^{-1}(y) & \text{if } y \in [0, g(0)] \\ 0 & \text{if } y \in [g(0), \infty] \end{cases}$$

Function g is called the additive generator of T.

In the following theorem we determine a class of t-norms in which the addition of fuzzy intervals is very simple [4]:

Theorem 3.1. *Let T be an Archimedean t-norm with additive generator g and let $A_i = (a_i^-, a_i^+, \alpha, \beta)_{LR}$ $i = 1, \ldots, n$ be fuzzy intervals of LR-type. If L and R are twice differentiable, concave functions, and if g is twice differentiable, strictly convex function then the membership function of the T-sum $S = A_1 + \ldots + A_n$ is*

$$S(z) = \begin{cases} 1 & \text{if } z \in [S^-, S^+] \\ g^{[-1]}\left(n \cdot g\left(L\left(\frac{S^- - z}{n \cdot \alpha}\right)\right)\right) & \text{if } z \in [S^- - n\alpha, S^-] \\ g^{[-1]}\left(n \cdot g\left(R\left(\frac{z - S^+}{n \cdot \beta}\right)\right)\right) & \text{if } z \in [S^+, S^+ + n\beta] \\ 0 & \text{otherwise} \end{cases} \tag{3.2}$$

where $S^- = a_1^- + \ldots + a_n^-$ and $S^+ = a_1^+ + \ldots + a_n^+$. □

Proof. It is clear that

$$\begin{aligned} S(z) &= \sup_{x_1+\ldots+x_n=z} T(A_1(x_1), \ldots, A_n(x_n)) = \\ &= \sup_{x_1+\ldots+x_n=z} g^{[-1]}(g(A_1(x_1)) + \ldots + g(A_n(x_n))) = \\ &= g^{[-1]}\Big(\inf_{x_1+\ldots+x_n=z} (g(A_1(x_1)) + \ldots + g(A_n(x_n)))\Big) \end{aligned} \tag{3.3}$$

It is also easy to see that the support of S is included in the interval $[S^- - n\alpha, S^+ + n\beta]$. From the decomposition rule of fuzzy intervals [2] it follows that the peak of S is $[S^-, S^+]$. Moreover, if we consider the right hand side of S (i.e. $S^+ \le z \le S^+ + n\beta$) then only the right hand sides of the terms A_i come into account in (3.3) (i.e. $a_i^+ \le x_i \le a_i^+ + \beta$, $i = 1, \ldots, n$). The same holds for the left hand side of S, this is why we deal in the following just with the right hand side of S.

So, let $S^+ \le z \le S^+ + n\beta$. The constraints

$$x_1 + \ldots + x_n = z \qquad a_i^+ \le x_i \le a_i^+ + \beta \qquad i = 1, \ldots, n$$

determine a compact and convex domain $\mathcal{K} \subset \mathbb{R}^n$ which can be considered as the section of the brick

$$\mathcal{B} := \left\{(x_1, \ldots, x_n) \;\middle|\; a_i^+ \le x_i \le a_i^+ + \beta \qquad i = 1, \ldots, n\right\}$$

by the hyperplane

$$\mathcal{P} := \left\{(x_1, \ldots, x_n) \;\middle|\; x_1 + \ldots + x_n = z\right\}$$

In order to calculate $S(z)$ we need to find the conditional minimum value of the function $\varphi : \mathcal{B} \to \mathbb{R}$

$$\varphi(x_1, \ldots, x_n) = g(A_1(x_1)) + \ldots + g(A_n(x_n))$$

subject to condition $(x_1, \ldots, x_n) \in \mathcal{K}$. We could change the infimum with minimum because $\mathcal{K}$ is compact and φ is continuous.

Following the Lagrange's multiplayers method it can be shown [4] that φ attains its conditional minimum at the point

$$\hat{x}_i = a_i^+ + \frac{z - S^+}{n} \qquad i = 1, \ldots, n$$

where

$$A_1(x_1) = \ldots = A_n(x_n)$$

This is the only stationary point of φ (i.e. where its partial derivatives vanish). This point is guaranteed to be a minimum by monotonocity and concavity of the shape function R and by monotonicity and strict convexity of the generator function g. Substituting the values $(\hat{x}_1, \ldots, \hat{x}_n)$ for $(x_1, \ldots, x_n)$ in (3.3) we immediatly get the desired result (3.2). ■

REFERENCES

[1] D. Dubois, H. Prade: Additions of interactive fuzzy numbers, *IEEE Transactions on Automatic Control*, Vol. AC-26, No.4. 926-936

[2] D.Dubois, H.Prade: Inverse Operations for Fuzzy Numbers, *Proc. of IFAC Symp. on Fuzzy Information, Knowledge, Representation and Decision Analysis*, Marseille, (1983) 399-404

[3] R. Fullér, T. Keresztfalvi: On Generalization of Nguyen's Theorem, *Fuzzy Sets and Systems*, **41**, (1991)

[4] R. Fullér, T. Keresztfalvi: t-norm based addition of fuzzy intervals, *Fuzzy Sets and Systems*, (submitted)

[5] H. T. Nguyen: A Note on the Extension Principle for Fuzzy Sets, *J. Math. Anal. Appl.*, 64 (1978) 369-380