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ON THE GENERALIZED METHOD-OF-CASE INFERENCE RULE*

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Abstract: We consider the generalized method-of-case (GMC) inference scheme with fuzzy antecedents, which has been introduced by Da in [1]. We show that when the fuzzy numbers involved in the observation part of the scheme have continuous membership functions; and the t-norm, t-conorm used in the definition of the membership function of the conclusion are continuous, then the conclusion defined by the compositional rule of inference depends continuously on the observation.

Keywords: Compositional rule of inference, triangular norm, triangular conorm, fuzzy implication operator, generalized method-of-case, stability

1. INTRODUCTION

The inference process from imprecise premises is becoming more and more important for knowledge-based systems, especially for fuzzy expert systems [5,7,8,9,11]. In Approximate Reasoning there are several kinds of fuzzy inference rules [9], e.g. entailment and conjunctive rules, the generalised modus ponens and tollens, the GMC. In this paper we will deal only with the GMC inference rule. When the predicates are crisp then the method of cases reads

$$\begin{array}{l} \quad A \text{ OR } B \\ \text{IF } A \text{ THEN } C \\ \text{IF } B \text{ THEN } C \\ \hline \qquad C \end{array}$$

This equivalent to saying that the formula $((A \vee B) \wedge (A \rightarrow C) \wedge (B \rightarrow C)) \rightarrow C$ is a tautology in binary logic where A, B and C are propositional variables. The proof of many theorems in conventional mathematics is based on this scheme,

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e.g. theorems involving the absolute value of a real variable are usually proved by considering separately positive and nonpositive values of the variable, and the conclusion is derived in each of these cases.

The goal of this paper is to investigate the effect of small changes of the observation to the conclusion of similar deduction schemes when the antecedents involve fuzzy concepts.

2. PRELIMINARIES

Let U be an ordinary nonvoid set. A fuzzy set [10] A of U is a mapping from U to [0,1]. The family of all fuzzy sets of U is denoted by $\mathcal{F}(U)$. A fuzzy set A with membership function $A : \mathbb{R} \to [0,1]$ is called fuzzy number [2] if A is normal, upper-semicontinuous, fuzzy convex and compactly supported. The fuzzy numbers will represent the continuous possibility distributions for fuzzy concepts. Let $(U, \|\cdot\|)$ be a normed space and let $A : U \to \mathbb{R}$ be a continuous function, then for any $\theta \geq 0$ we define $\omega_A(\theta)$, the modulus of continuity of A, as

$$\omega_A(\theta) = \max_{\|u-v\|\leq\theta} |A(u) - A(v)|.$$

An α-level set of a fuzzy number A is a non-fuzzy set denoted by $[A]^\alpha$ and is defined by $[A]^\alpha = \{t \in \mathbb{R} | A(t) \geq \alpha\}$ for $\alpha \in (0,1]$ and $[A]^\alpha = \text{cl}(\text{supp}A)$ if $\alpha = 0$, where $\text{cl}(\text{supp}A)$ denotes the closure of the support of A.

We metricize the set of fuzzy numbers by the metric [4]

$$D(A,B) = \sup_{\alpha\in[0,1]} \max_{i=1,2}\{|a_i(\alpha) - b_i(\alpha)|\},$$

where $[A]^\alpha = [a_1(\alpha), a_2(\alpha)]$, $[B]^\alpha = [b_1(\alpha), b_2(\alpha)]$, $\alpha \in [0,1]$.

Recall that a function $T : [0,1] \times [0,1] \to [0,1]$ is said to be a triangular norm [6] (t-norm for short) iff T is symmetric, associative, non-decreasing in each argument, and $T(x,1) = x$ for all $x \in [0,1]$.

A function $S : [0,1] \times [0,1] \to [0,1]$ is called triangular conorm (t-conorm for short) iff S is symmetric, associative, non-decreasing in each argument, and $T(x,0) = x$ for all $x \in [0,1]$.

In the sequel we need the following lemma [3].

Lemma 1. *Let $\delta \geq 0$ be a real number and let A, B be fuzzy numbers. If $D(A,B) \leq \delta$, then*

$$\sup_{t\in\mathbb{R}} |A(t) - B(t)| \leq \max\{\omega_A(\delta), \omega_B(\delta)\}$$

where $[A]^\alpha = [a_1(\alpha), a_2(\alpha)]$, $[B]^\alpha = [b_1(\alpha), b_2(\alpha)]$, $\alpha \in [0,1]$. □

3. THE GENERALIZED FUZZY METHOD OF CASES

Let X, Y and Z be variables taking values in universes U, V and W, respectively and let $A, A' \in \mathcal{F}(U)$, $B, B' \in \mathcal{F}(V)$ and $C' \in \mathcal{F}(W)$, then the generalized method of cases reads:

Observation: X is A' OR Y is B'
Antecedent 1: IF X is A THEN Z is C
Antecedent 2: IF Y is B THEN Z is C

Conclusion: Z is C'

The conclusion C' is given by applying the general compositional rule of inference

$$C'(w) = \sup_{(u,v)\in U\times V} T(S(A'(u), B'(v)), I(A(u), C(w)), I(B(v), C(w))) \quad (1)$$

where T is an arbitrary triangular norm, S is an arbitrary conorm and I represents an arbitrary fuzzy implication operator [5].
For instance,

Observation: This bunch of grapes is fairly sweet OR this bunch of grapes is more or less yellow
Antecedent 1: IF a bunch of grapes is yellow THEN the bunch of grapes is ripe
Antecedent 2: IF a bunch of grapes is sweet THEN the bunch of grapes is ripe

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Conclusion: This bunch of grapes is more or less ripe

Consider now the generalized method-of-case scheme with different fuzzy observations A', A'', B', B'':

$$\begin{array}{lcl} X \text{ is } A' \text{ OR } Y \text{ is } B' & \quad & X \text{ is } A'' \text{ OR } Y \text{ is } B'' \\ \text{IF } X \text{ is } A \text{ THEN } Z \text{ is } C & & \text{IF } X \text{ is } A \text{ THEN } Z \text{ is } C \\ \text{IF } Y \text{ is } B \text{ THEN } Z \text{ is } C & & \text{IF } Y \text{ is } B \text{ THEN } Z \text{ is } C \\ \hline Z \text{ is } C' & & Z \text{ is } C'' \end{array} \quad (2)$$

where C' and C'' are defined by the compositional rule of inference, in the sense of (1), i.e.

$$C'(w) = \sup_{(u,v)\in U\times V} T(S(A'(u), B'(v)), I(A(u), C(w)), I(B(v), C(w))) \quad (3)$$

$$C''(w) = \sup_{(u,v)\in U\times V} T(S(A''(u), B''(v)), I(A(u), C(w)), I(B(v), C(w))) \quad (4)$$

The following theorem gives an upper estimation for the distance between the conclusions C' and C'' obtained from GMC schemes (2).

Theorem 1. *Let T and S be continuous functions and let A', A'', B' and B'' be continuous fuzzy numbers. Then with the notation*

$$\triangle = \max\{\omega_{A'}(D(A', A'')), \omega_{A''}(D(A', A'')), \omega_{B'}(D(B', B'')), \omega_{B''}(D(B', B''), \}$$

we have

$$\sup_{w\in W} |C'(w) - C''(w)| \le \omega_T(\omega_S(\triangle)), \quad (5)$$

where $\triangle = \max\{\omega_{A'}(\delta), \omega_{A''}(\delta), \omega_{B'}(\delta), \omega_{B''}, \}$, and the conclusions C, C' are defined by (3) and (4), respectively. □

Proof. Using Lemma 1 and the monotinicity of the modulus of continuity we get

$$\begin{aligned}
&|C'(w) - C''(w)| = \\
&\quad |\sup_{(u,v)\in U\times V} T(S(A'(u), B'(v)), I(A(u), C(w)), I(B(v), C(w))) - \\
&\quad \sup_{(u,v)\in U\times V} T(S(A''(u), B''(v)), I(A(u), C(w)), I(B(v), C(w)))| \le \\
&\quad \sup_{(u,v)\in U\times V} |T(S(A'(u), B'(v)), I(A(u), C(w)), I(B(v), C(w))) - \\
&\qquad T(S(A''(u), B''(v)), I(A(u), C(w)), I(B(v), C(w))) \le \\
&\quad \sup_{(u,v)\in U\times V} \omega_T\Big(|S(A'(u), B'(v)) - S(A''(u), B''(v))|\Big) \le \\
&\quad \sup_{(u,v)\in U\times V} \omega_T\Big(\omega_S \|(A'(u), B'(v)) - (A''(u), B''(v))\|)\Big) = \\
&\quad \sup_{(u,v)\in U\times V} \omega_T\Big(\omega_S(\max\{|A'(u) - A''(u)|, |B'(v) - B''(v)|\})\Big) \le \\
&\qquad \omega_T(\omega(S(\triangle)),
\end{aligned}$$

which proves the theorem. ■

It should be noted that (i) from (5) it follows that $C' \to C''$ uniformly as $\triangle \to 0$, which means the stability (in the classical sense) of the conclusion under small changes of the fuzsy terms; (ii) the stability or instability of the conclusion does not depend on the implication operator I.

For illustration of this theorem consider the following schemes with arbitrary continuous fuzzy numbers A, B and C:

$$\begin{array}{lcl} X \text{ is } A \text{ OR } Y \text{ is } B & \quad & X \text{ is } A \text{ OR } Y \text{ is more or less } B \\ \text{IF } X \text{ is } A \text{ THEN } Z \text{ is } C & & \text{IF } X \text{ is } A \text{ THEN } Z \text{ is } C \\ \text{IF } Y \text{ is } B \text{ THEN } Z \text{ is } C & & \text{IF } Y \text{ is } B \text{ THEN } Z \text{ is } C \\ \hline Z \text{ is } C' & & Z \text{ is } C'' \end{array} \qquad (2)$$

where (more or less B) $(y) := \sqrt{B(y)}$, $y \in I\!R$; $T(u,v) = u \wedge v$, (minimum norm); $S(x,y) = x \vee y$ (maximum conorm); $I(x,y) = 1$ if $x \leq y$, and $I(x,y) = y$ if $y < x$ (Gödel's implication operator).
Following Da [1, p.125], we get $C' = C$ and $C'' =$ more or less C, i.e.

$$C''(w) = \sqrt{C(w)}, \quad w \in I\!R.$$

So,

$$\sup_{w \in I\!R} |C'(w) - C''| = \sup_{w \in I\!R} |C(w) - \sqrt{C(w)}| = 1/4.$$

On the other hand, using the relationships,

$$D(A, A) = 0, \quad D(B, \text{ more or less } B) \leq 1/4; \quad \omega_S(\Delta) = \Delta, \quad \omega_T(\Delta) = \Delta, \quad \Delta > 0;$$

Theorem 1 gives

$$\sup_{w \in I\!R} |C'(w) - C''(w)| \leq \max\{\omega_B(1/4), \ \omega_{\text{more or less } B}(1/4)\} \leq 1/4,$$

which means, that our estimation (5) is sharp, i.e. there exist C' and C'', such that

$$\sup_{w \in I\!R} |C'(w) - C''(w)| = \omega_T(\omega_S(\Delta)).$$

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