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AGGREGATION OF PREFERENCES: AN AXIOMATIC APPROACH WITH APPLICATIONS

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Abstract: In this paper an axiomatic justification of using generalized arithmetic means as aggregation rules is presented. Necessary and sufficient conditions are given in order to guarantee that the aggregated preference is reflexive, T-transitive and S-complete. Finally we investigate the existence of a maximal nondominated alternative in a finite set.

Keywords: Multi-criteria decision making, aggregation, generalized means, preference structures, nondominated alternatives

1. INTRODUCTION

Assume that A is a set of alternatives, $c_1, \dots, c_m$ are m criteria defined by m fuzzy preference relations $R_1, \dots, R_m$ on A, i.e. $R_i : A^2 \to [0,1]$ are functions such that for any $a, b \in A$, $R_i(a,b)$ is the truth value of the statement

"a is not worse than b, according to c_i".

A key problem of multicriteria decision-making consists of aggregating preferences $R_1, \dots, R_m$ so that the aggregated relation R should reflect, in a sense, all of the criteria and at the same time, R should enable us to select the 'best' alternative from A.

Two previous results can be mentioned on aggregation of preferences. The so-called 'aggregative operator' introduced by Dombi [1] is based on the general solution of a functional equation related to associativity. An alternative approach was given by Dyckhoff [2]: autodistributivity was considered instead of associativity.

In this paper we propose a third approach based on a result of Kolmogoroff [5]. Our final conclusion is the same as that of Dyckhoff [2]: the aggregated preference R should have the following form:

$$R(a,b) = \psi^{-1}[\sum_{i=1}^{m} \lambda_i \psi(R_i(a,b))],$$

where $\lambda_1, \ldots, \lambda_m$ are the relative importances of criteria $c_1, \ldots, c_m$, respectively, and ψ is an automorphism of the unit interval.

In a very common and important case all preferences R_i are reflexive, transitive and complete crisp relations. Even in this situation the aggregated preference R is a fuzzy binary relation. So assume that fuzzy set-theoretical operations are modelled by a De Morgan triple (T, S, n). The results of Ovchinnikov and Roubens [7], [8] and Fodor [3] imply that there exists an automorphism ϕ of the unit interval such that

$$\begin{aligned} T(x,y) &= T^{\phi}(x,y) = \phi^{-1}[\max\{\phi(x) + \phi(y) - 1, 0\}], \\ S(x,y) &= S^{\phi}(x,y) = \phi^{-1}[\min\{\phi(x) + \phi(y), 1\}], \\ n(x) &= n^{\phi}(x) \quad = \phi^{-1}[1 - \phi(x)]. \end{aligned}$$

Denote η the composition of ϕ and ψ^{-1}. Obviously, η is also an automorphism of [0,1]. We give necessary and sufficient conditions on η under that the aggregated preference R is reflexive, T-transitive and S-complete fuzzy relation.

Finally, applying these results, we obtain sufficient conditions on R_i's and η in order to guarantee the existence of a maximal nondominated element in a finite A.

In this paper only the results are presented. For proofs see [4].

2. AGGREGATION OF PREFERENCES

Let A be a set of alternatives, $c_1, \ldots, c_m$ m criteria defined by m fuzzy preference relations $R_1, \ldots, R_m$, where $R_i(a,b)$ is the truth value of the statement

"a is not worse than b, according to c_i".

Denote $\lambda_1, \ldots, \lambda_m$ the relative importance of $c_1, \ldots, c_m$, respectively. This means that $\lambda_i > 0$ and $\sum_{i=1}^{m} \lambda_i = 1$.

We distinguish three cases:

a) All criteria are equally important (i.e., $\lambda_1 = \ldots = \lambda_m = 1/m$).

b) All λ_i's are rational numbers.

c) All λ_i's are arbitrary real numbers.
The aggregated preference is denoted by R^m.

Case a: Equally important criteria.

In this case the following axioms seems to be natural:

A1. $R^m(a,b)$ depends only on the values $R_1(a,b),\dots,R_m(a,b)$.

A2. If $R_1(a,b) = \dots = R_m(a,b)$ then $R^m(a,b) = R_1(a,b)$.

A3. Aggregating $R_1(a,b),\dots,R_k(a,b), R_{k+1}(a,b),\dots,R_m(a,b)$ we obtain $R^m(a,b)$. Now aggregate $R_1(a,b),\dots,R_k(a,b)$, the result is $R^k(a,b)$. Then substitute each R_i by $R^k (i = 1,\dots,k)$:
$R^k(a,b),\dots,R^k(a,b), R^{k+1}(a,b),\dots,R_m(a,b)$.
Aggregate these new values; the resulted preference is R^m_*. Then let $R^m(a,b) = R^m_*(a,b)$.

A4. If $a,b,c,d \in A$ are such that $R_i(a,b) = R_i(c,d)$ for $1 \le i \le m-1$ and $R_m(a,b) < R_m(c,d)$ then let $R^m(a,b) < R^m(c,d)$.

A5. $R^m(a,b)$ depends continuously on the values $R_1(a,b),\dots,R_m(a,b)$.

A6. $R^m(a,b)$ is invariant under permutations of $R_1,\dots,R_m$.
Translating these axioms we get the following conditions:

B1. For every $m \ge 2$ integer there exists $M_m : [0,1]^m \to [0,1]$ such that $R^m(a,b) = M_m(R_1(a,b),\dots,R_m(a,b))$.

B2. $M(x,x,\dots,x) = x$ for every $x \in [0,1]$.

B3. $x_1,\dots,x_k,\ x_{k+1},\dots,x_m$; If $x = M_k(x_1,\dots,x_k)$ then $M(x_1,\dots,x_m) = M(x,\dots,x,x_{k+1},\dots,x_m)$

B4. $x_m < y_m$ implies $M(x_1,\dots,x_{m-1},x_m) < M(x_1,\dots,x_{m-1},y_m)$.

B5. M is continuous.

B6. Let $(i_1,\dots,i_m)$ be a permutation of $(1,\dots,m)$. Then $M(x_1,\dots,x_m) = M(x_{i_1},\dots,x_{i_m})$.

Theorem 2.1. ([5]) *M fulfils conditions B1-B6 if and only if there exists a continuous, increasing function $f : \mathbb{R} \to \mathbb{R}$ such that*

$$M_m(x_1,\dots,x_m) = f^{-1}\left\{\frac{f(x_1)+\dots+f(x_m)}{m}\right\}. \qquad \square$$

Case b: All λ_i's are rational numbers.

Let us denote $M_{m+n}(x_1, \dots, x_m, y_1, \dots, y_n)$ by $M(mx, ny)$ for short if $x_1 = \dots = x_m = x, y_1 = \dots = y_n = y$ and all λ_i's are equal. Then for any positive integer p we have

$$M(pmx, pny) = M(pM(mx, ny)) = M(mx, ny),$$

thus $mn' = nm'$ implies that

$$M(mx, ny) = M(mn'x, nn'y) = M(nm'x, nn'y) = M(m'x, n'y). \tag{2.1}$$

Assume now that $\lambda_1, \dots, \lambda_m$ are positive rational numbers with $\sum_{i=1}^{m} \lambda_i = 1$; where λ_i means the relative importance of criterion c_i. In this case, according to (2.1), it seems to be reasonable to define $M_\lambda(x_1, \dots, x_m)$ as follows. Let $p_1, \dots, p_m, q$ be positive integers such that $\lambda_i = p_i/q$. Then let $\lambda = (\lambda_1, \dots, \lambda_m)$ and

$$M_\lambda(x_1, x_2, \dots, x_m) = M(p_1x_1, p_2x_2, \dots, p_mx_m), \tag{2.2}$$

where M is any function which fulfils B1-B6. Obviously, M_λ satisfies conditions B1-B5. Using Theorem 2.1 we can obtain the following representation for M_λ.

Theorem 2.2. *Assume that $\lambda_1, \dots, \lambda_m$ are rational numbers. Then M_λ fulfils axioms B1-B5 if and only if the corresponding M fulfils B1-B6, i.e., if and only if there exists a continuous, increasing function $f : \mathbb{R} \to \mathbb{R}$ such that*

$$M_\lambda(x_1, \dots, x_m) = f^{-1}\left\{\sum_{i=1}^{m} \lambda_i f(x_i)\right\}. \tag{2.3}$$

□

Case c: All λ_i's are arbitrary real numbers.

We can obtain representation (2.3) by approximating irrational λ_k's via rational series.

Assume that $M_\lambda(x_1, \dots, x_m)$ fulfils conditions B1-B5 and let

$$m_\lambda^i(x) = M_\lambda(0, \dots, 0, x, 0, \dots, 0), \quad i = 1, \dots, m,$$

where the only nonzero element x is on the ith place.

We can classify generalized means represented by (2.3) as follows

(a) $m_\lambda^i(x) = 0$ for every $j = 1, \dots, m$ and $x > 0$, (2.4)

(b) $m_\lambda^i(x) > 0$,for every $j = 1, \dots, m$ and $x > 0$, (2.5)

Since values of relations lie in the unit interval, the following theorem can be proved by using representation (2.3).

Theorem 2.3. *Condition* (2.4) *holds if and only if there exists an automorphism* ψ *of* $[0,1]$ *such that*

$$M_\lambda(x_1,\ldots,x_m) = \psi^{-1}[(\psi(x_1)^{\lambda_1}\psi(x_2)^{\lambda_2}\ldots\psi(x_m)^{\lambda_m}]. \tag{2.6}$$

□

Theorem 2.4. *Condition* (2.5) *holds if and only if there exists an automporhism* ψ *of the unit interval such that*

$$M_\lambda(x_1,\ldots,x_m) = \psi^{-1}\left(\sum_{i=1}^{m}\lambda_i\psi(x_i)\right). \tag{2.7}$$

□

It is reasonable to assume that we are dealing with an M_λ for which (2.5) is true. Otherwise we would obtain a strange kind of aggregation : $R(a,b)$ could be zero even if $R_1(a,b)$ is close to $1 (R_2(a,b) = 0)$. So in the rest of the paper we assume that M_λ fulfils condition (2.5), i.e. M_λ has the representation (2.7).

3. FUZZY PREFERENCE RELATIONS AND STUCTURES

Assume that we have m crisp preference relations $R_1,\ldots,R_m$ representing m criteria $c_1,\ldots,c_m$. Even in this situation, according to results of Section 2, the aggregated preference R is a valued relation. On the other hand, we need strict preference relation P (based on R) to investigate maximal nondominated elements of the set of alternatives A. Thus we need some notions and results from the theory of fuzzy sets and preference modelling.

Assume that (T,S,n) is a De Morgan triple for modelling intersection, union and complementation, respectively. Let R be any fuzzy binary relation on A. Then R is called

- *reflexive* if $R(a,a) = 1$ for every $a \in A$,
- *T-transitive* if $T(R(a,b),R(b,c)) \leq R(a,c)$ for every $a,b,c \in A$,
- *S-complete* if $S(R(a,b),R(b,a)) = 1$ for every $a,b \in A$.

In a very common situation, crisp preferences R_i are reflexive, transitive and complete binary relations. Let

$$R(a,b) = \psi^{-1}\left(\sum_{i=1}^{m}\lambda_i\psi(R_i(a,b))\right). \tag{3.1}$$

According to the results of Ovchinnikov and Roubens [7,8] and Fodor [3], there exists an automorphism ϕ of the unit interval such that

$$\begin{aligned} T(x,y) &= T^{\phi}(x,y) = \phi^{-1}[\max\{\phi(x)+\phi(y)-1,0\}], \\ S(x,y) &= S^{\phi}(x,y) = \phi^{-1}[\min\{\phi(x)+\phi(y),1\}], \\ n(x) &= n^{\phi}(x) \quad = \phi^{-1}(1-\phi(x)). \end{aligned}$$

Thus we have two automorphisms of the unit interval: ψ in the aggregation and ϕ in the set-theoretical operations. Let $\eta(x) = \phi o \psi^{-1}(x)$. Obviously, η is also an automorphism of $[0,1]$. In the following theorems we give necessary and sufficient conditions on reflexivity, T-transitivity and S-completeness of R. Let $T_0(x,y) = \max\{x+y-1,0\}$.

Theorem 3.1. *R is reflexive if and only if all R_i are reflexive binary relations $(i = 1,\dots,m)$.* □

Theorem 3.2. *Assume that R_i is transitive for $i = 1,\dots,m$. Then R is T^{ϕ}-transitive if and only if*

$$T^{\eta}(x,y) \le T_0(x,y). \tag{3.2}$$

□

Theorem 3.3. *Suppose that R_i is complete for $i = 1,\dots,m$. Then R is S^{ϕ}-complete if and only if*

$$x + y \ge 1 \text{ implies } \eta(x) + \eta(y) \ge 1. \tag{3.3}$$

□

Example 3.4. Let $\eta_p(x) = \frac{x^p}{x^p+(1-x)^p} (p > 0)$. Then (3.2) is satisfied by any η_p. On the other hand, η_p fulfils (3.3) if and only $0 < p \le 1$.

4. NONDOMINATED ALTERNATIVES

In this section we assume that A is finite. Let R be defined by (3.1). We need also fuzzy strict preference P, associated with R, in order to investigate nondominated alternatives (see Orlovski [6]), where for $a, b \in A$ $P(a,b)$ means the truth value of the statement

"a is better than b".

Then

$$\mu_{ND}(a) = \min_{b \in A} N[P(b,a)] \tag{4.1}$$

is the *fuzzy set of nondominated elements* of A, or in other words, $\mu^{ND}(a)$ is the truth value of the statement "there is no $b \in A$ better than a". Finally, let

$$A^{ND} = \{a \in A;\ \mu^{ND}(a) = \max_{c \in A} \mu^{ND}(c) > 0\}$$

be the (crisp) set of *maximal nondominated elements* of A.

According to recent axiomatic approaches to the definition of P, proposed by Ovchinnikov and Roubens [8] and Fodor [3], let

$$P(a,b) = \phi^{-1}[\max\{\phi(R(a,b)) - \phi(R(b,a)), 0\}],$$

where ϕ is the same automorphism as in $(T^{\phi}, S^{\phi}, n^{\phi})$. Then the following results can be proved.

Theorem 4.1. *If R is T^{ϕ}-transitive and S^{ϕ}-complete then A^{ND} is not empty.* □

Theorem 4.2. *If η is such that $\eta(x) + \eta(1-x) = 1$ and $T^{\eta}(x,y) \le T_0(x,y)$ then there exists a maximal nondominated element in A.* □

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