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STUDY OF L-FUZZY SIMILARITIES

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Abstract: Here we define the concept and properties of L-fuzzy relations on a fuzzy set A of a set X. Then we study the properties of composition of these relations. The algebraic structure of L-fuzzy relations, specially similarities are also studied.

Keywords: L-fuzzy relation, relation composition, complete lattice

1. INTRODUCTION

Let X be a set, $L = (L, \wedge, \vee, 0, 1) = (L, \leq)$ a complete, completely distributive lattice. A map $A : X \longrightarrow L$ is called a L-fuzzy set on X, whose family is denoted by $F(X)$, which is again a complete, completely distributive lattice under the operations defined by the operations of L. The Cartesian product of $A \in F(X)$ and $B \in F(Y)$:

$$(A \times B)(x, y) = A(x) \wedge B(y), \ \forall (x, y) \in X \times Y.$$

An $R \leq A \times B \in F(X \times Y)$ is said to be a L-fuzzy relation on $A \times B$. If $X = Y$ and $A = B$ we speak about a L-fuzzy relation on A. In this case we define the diagonal relation D_A as follows:

$$D_A(x, y) = \begin{cases} A(x), & \text{if } x = y \\ 0, & \text{otherwise} \end{cases} \qquad \forall (x, y) \in X \times X.$$

The converse R^{-1} of on $R \leq A \times B \in F(X \times Y)$:

$$R^{-1}(x, y) = R(y, x), \ \ \forall x, y \in Y \times X.$$

The product of L-fuzzy relations $R \leq A \times B \in F(X \times Y)$ and $Q \leq B \times C \in F(Y \times Z)$ is defined by

$$(R \circ Q)(x, z) = \vee_{y \in Y} (R(x, y) \wedge Q(y, z)), \ \ \forall (x, z) \in X \times Z.$$

A L-fuzzy relation R on $A \in F(X)$ is called: a) reflexive, if $D_A \leq R$; b) symmetric, if $R^{-1} = R$; c) transitive, if $R \circ R \leq R$. Such an R is said to be a L-fuzzy similarity on $A \in F(X)$. Denote by $S(A)$ their family, and by $R(A)$ the family of all L-fuzzy relations on A.

2. RESULTS

Theorem 1. *$(R(A), \leq)$ is a complete lattice.* □

Proof. Let $\{R_i\}_{i \in I}$ be an arbitrary (nonvoid) family from $R(A)$. Define the L-fuzzy relation R as follows.

$$R(x,y) \leq \inf_{i \in I} R_i(x,y), \quad \forall x, y \in X.$$

Since L is a complete lattice, hence R is well defined. By the definition of infimum we have

$$R(x,y) \leq R_i(x,y), \quad \forall i \in I, \quad \forall x, y \in X.$$

Moreover, from this we get:

$$R(x,y) \leq R_i(x,y) \leq A(x) \wedge A(y), \quad \forall x, y \in X,$$

that is $R \in R(A)$ follows. Hence R is a lower bound of the family $\{R_i\}_{i \in I}$ in $R(A)$.

Now let $Q \in R(A)$ be such that $Q \leq R_i$ for all $i \in I$. Then also by the definition of infimum

$$Q(x,y) \leq \inf_{i \in I} R_i(x,y) = R(x,y), \quad \forall x, y \in X,$$

i.e. $Q \leq R$. This proves that R is the greatest lower bound of the family $\{R_i\}_{i \in I}$ from $R(A)$, in other words

$$R = \inf_{i \in I} R_i.$$

Hence by [2,p.21] $R(A)$ is a complete lattice. ■

Theorem 2. *$S(A)$ is a closure system on $A \times A \in F(X \times X)$.* □

Proof. Let $\{S_i\}_{i \in I}$ be any family from $S(A)$. Obviously $S_i \in R(A)$ for each $i \in I$, that is

$$S_i(x,y) \leq A(x) \wedge A(y), \quad \forall i \in I, \quad \forall x, y \in X.$$

Since by Theorem 1 $R(A)$ is a complete lattice, consequently

$$(\wedge_{i \in I} S_i)(x,y) = \wedge_{i \in I} S_i(x,y) \leq A(x) \wedge A(y), \quad \forall x, y \in X.$$

that is $\wedge_{i\in I}S_i = \inf_{i\in I} S_i = S \in R(A)$.

Furthermore, from the reflexivity of each S_i there follows

$$S(x,y) = \wedge_{i\in I}S_i(x,y) \geq \wedge_{i\in I}D_A(x,y) = D_A(x,y) \quad \forall x,y \in X,$$

i.e. $S \geq D_A$, which proves the reflexivity of S.

The symmetry of S easily follows from the equality (using the symmetry of each S_i):

$$\wedge_{i\in I}S_i(x,y) = \wedge_{i\in I}S_i(y,x), \quad \forall x,y \in X.$$

To show the transitivity of S we write

$$\begin{aligned}(S \circ S)(x,y) &= (\wedge_{i\in I}S_i \circ \wedge_{i\in I}S_i(x,y) = \\ &= \vee_{z\in X}((\wedge_{i\in I}S_i(x,z) \wedge (\wedge_{i\in I}S_i)(z,y)) = \\ &= \wedge_{i\in I}S_i(\vee_{z\in X})S_i(x,z) \wedge S_i(z,y) = \\ &= \wedge_{i\in I}(S_i \circ S_i)(x,y)) \leq \wedge_{i\in I}S_i(x,y) = \\ &= (\wedge_{i\in I}S_i)(x,y) = S(x,y), \quad \forall x,y \in X,\end{aligned}$$

that is $S \circ S \leq S$ which was to be proved. ■

We have showed that the meet of any family $\{S_i\}_{i\in I}$ of L-fuzzy similarities on $A \in F(X)$, so $S(A)$ is really a closure system on $A \times A$.

Proposition 1. *$A \times A$ is an L-fuzzy similarity on $A \in F(X)$.* □

Proof. The statement $A \times A \in S(A)$ is an immediate consequence of Theorem 2: the base set is always a member of any closure system on it. ■

Theorem 3. *$(S(A), \leq)$ is a complete lattice.* □

Proof. Consequence of Theorem 2. See e.g. [1,p.35]. ■

Theorem 4. *The composition of L-fuzzy relations is well defined, i.e. if*

$$R \leq A \times B \in F(X \times Y) \quad \text{and} \quad Q \leq B \times C \in F(Y \times Z),$$

then

$$R \circ Q \leq A \times C \in F(X \times Z).$$ □

Proof. By definition

$$R(x,y) \leq A(x) \wedge B(y), \quad \forall (x,y) \in X \times Y,$$
$$Q(z,z) \leq B(y) \wedge C(z), \quad \forall (y,z) \in Y \times Z.$$

From these by the monotonicity of $\wedge$ we get.

$$\begin{aligned}R(x,y)\wedge Q(y,z) &\le (A(x)\wedge B(y))\wedge(B(y)\wedge C(z)) = \\ &= A(x)\wedge B(y)\wedge C(z)\le A(x)\wedge C(z)\wedge 1 = \\ &= A(x)\wedge C(z)\end{aligned}$$

for all $(x,y,z)\in X\times Y\times Z$. Consequently it is also true that

$$\vee_{y\in Y}(R(x,y)\wedge Q(y,z)) = (R\circ Q)(x,z)\le A(x)\wedge C(z)$$

for all $(x,y)\in X\times Z$, which was to be proved. ∎

Theorem 5. *The composition of L-fuzzy relations is associative.* □

Proof. Let $R\le A\times B\in F(X\times Y)$, $Q\le B\times C\in F(Y\times Z)$ and $T\le C\times D\in F(Z\times W)$. By Theorem 4 $R\circ Q\le A\times C\in F(X\times Z)$, $(R\circ Q)\circ T\le A\times D\in F(X\times W)$. Similarly: $Q\circ T\le B\times D\in F(Y\times W)$, $R\circ(Q\circ T)\le A\times D\in F(X\times W)$.

Now let (x,w) be any element of $X\times W$, and write

$$\begin{aligned}((R\circ Q)\circ T)(x,w) &= \vee_{z\in Z}((R\circ Q)(x,z)\wedge T(z,w)) = \\ &= \vee_{z\in Z}(\vee_{y\in Y}(R(x,y)\wedge Q(y,z))\wedge T(z,w))) = \\ &= \vee_{z\in Z}\ \vee_{y\in Y}\ ((R(x,y)\wedge Q(y,z))\wedge T(z,w)) = \\ &= \vee_{y\in Y}(R(x,y)\wedge \vee_{z\in Z}(Q(y,z)\wedge T(z,w))) = \\ &= \vee_{y\in Y}(R(x,y)\wedge (Q\circ T)(y,w)) = \\ &= (R\circ(Q\circ T))(y,w).\end{aligned}$$

∎

Theorem 6. *The composition of L-fuzzy relations on $A\in F(X)$ is isotone, i.e. if $R,Q,T\in R(A)$ with $R\le Q$, then $R\circ T\le Q\circ T$ and $T\circ R\le T\circ Q$.* □

Proof. By Theorem 4 $R\circ T$, $Q\circ T$, $T\circ R$, $T\circ Q$ all are elements of $R(A)$. Let (x,y) be an arbitrary element of $X\times X$. Then for any $z\in X$ by our assumption $R(x,z)\le Q(z,y)$. Moreover, by the monotonicity of meet in L we have

$$R(x,y)\wedge T(z,y)\le Q(x,z)\wedge T(z,y)$$

for all $z\in X$, consequently for their union, too,

$$\vee_{z\in X}(R(x,z)\wedge T(z,y))\le \vee_{z\in X}(Q(x,z)\wedge T(z,y)).$$

But the left hand side here is $(R \circ T)(x, y)$, and the right hand side is $(Q \circ T)(x, y)$ for any $(x, y) \in X \times X$. Thus we get the desired result:

$$R \circ T \leq Q \circ T.$$

The other statement can be proved similarly. ■

Theorem 7. *$(R(A), \leq, \circ)$ is a lattice ordered monoid.* □

Proof. Considering Theorems 1, 4, 5 and 6, it suffices to show that D_A is the identity for $\circ$. Really, for any $x, y \in X$ and any $R \in R(A)$:

$$\begin{aligned}(R \circ D_A(x,y) &= \vee_{z\in X}(R(x,z) \wedge D_A(z,y)) = \\ &= \begin{cases} R(x,y) \wedge A(y), & \text{if } z = y \\ \vee_{z\in X}(R(x,z) \wedge 0) = 0, & \text{if } z \neq y \end{cases} = \\ &= R(x,y) \wedge A(y) = R(x,y),\end{aligned}$$

since $R(x, y) \leq A(x) \wedge A(y) \leq 1 \wedge A(y) = A(y)$ by definition. We can analogously show that $(D_A \circ R)(x, y) = R(x, y)$ for all $x, y \in X$. Thus $R \circ D_A = D_A \circ R$, what we wanted to show. ■

Proposition 2. *If $R \in R(A)$ is reflexive, then $R(x, x) \geq R(x, y)$ for all $x, y \in X$.* □

Proof. $R(x, y) \leq A(x) \wedge A(y) \leq A(x) \wedge 1 = A(x) = R(x, x)$, $\forall x, y \in X$. ■

Theorem 8. *The L-fuzzy similarities are idempotent element in the monoid $(R(A), \circ)$.* □

Proof. Let x, y be arbitrary elements of X, and let $S \in S(A)$. Then

$$\begin{aligned}(S \circ S)(x,y) &= \vee_{z\in X}(S(x,z) \wedge S(z,y)) \geq S(x,x) \wedge S(x,y) = \\ &= S(x,y)\end{aligned}$$

by Proposition 2. But by transitivity

$$(S \circ S)(x, y) \leq S(x, y)$$

is also true any $x, y \in X$. Thus really $S \circ S = S$, as stated. ■

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