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MEASURES OF FUZZINESS

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Abstract: First in this paper we give a brief summary about the various kinds of fuzziness measures investigated so far and formulate some critical aspects of the theory. Then we propose a general construction by means of the multivalent logical operators which makes a unified treatment possible for the problem how fuzzy a fuzzy set is. Finally we discuss our results and hypotheses on the fuzziness measures of multivalent logical expressions.

Keywords: Measures of fuzziness, fuzzy operators, multivalent logic

1. INTRODUCTION

In 1972 DeLuca and Termini introduced the concept of fuzziness measure in order to obtain a global measure of the indefiniteness connected with the situations described by fuzzy sets. Such a measure characterizes the sharpness of the membership functions. It also can be regarded as an entropy in the sense that it measures the uncertainty about the presence or absence of a certain property over the investigated set. Several authors have proposed scalar indices to measure the degree of fuzziness of a fuzzy set.

Let X be an ordinary nonempty, finite set. Then a fuzzy set over X is a map $f : X \to [0,1]$. Such a f is a generalization of the characteristic function used in ordinary set theory. In this paper we consider only finite sets $X = \{x_1, x_2, \ldots, x_N\}$. A measure of fuzziness is a nonnegative function d on the set of fuzzy sets over X, that is

$$d : [0,1]^X \to [0,\infty), \quad \text{where } [0,1]^X := \{f \mid f : X \to [0,1]\}.$$

Since the fuzziness of f arises from the presence of membership values different from 0 and 1, $d(f)$ has to be determined by these function values. That is

$$d(f) = F_N(f(x_1), f(x_2), \ldots, f(x_N))$$

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for some symmetric function $F_N : [0,1]^N \to [0,\infty)$ $(N = 1, 2, \ldots)$.
We use the following basic concepts in this paper:

1. If $f, g \in [0,1]^X$, then the join and union lattice operations are defined by $(f \cap g)(x) = min\{f(x), g(x)\}$ and $(f \cup g)(x) = max\{f(x), g(x)\}$
2. If $f \in [0,1]^X$ and $g \in [0,1]^Y$, then the $f \times g \in [0,1]^{X \times Y}$ direct product is defined by $(f \times g)(x,y) := f(x) \cdot g(y)$, $x \in X$, $y \in Y$.
3. If $f \in [0,1]^X$, then the pseudo-complement of f is defined by $f^-(x); = 1 - f(x)$, $x \in X$.
4. If $f \in [0,1]^X$, then power of f is defined by $P(f) := \sum_{x \in X} f(x)$.
5. If $f \in [0,1]^X$, then the $\frac{1}{2}$-cut of f is defined by:

$$f_{\frac{1}{2}}(x) = 0, \quad \text{if} \quad f(x) < \frac{1}{2},$$

$$f_{\frac{1}{2}}(x) = 1, \quad \text{if} \quad f(x) \geq \frac{1}{2}.$$

We give a list of desirable properties for a measure of fuzziness. Various combinations of these can be found in the literature, because different authors demand different properties for measure d.

(P1) *Sharpness:* $d(f) = 0$ iff $f(x) \in \{0, 1\}$ for all $x \in X$ (i.e. f is "sharp").

(P2) *Maximality:* $d(f)$ is maximum iff $f(X) = \{\frac{1}{2}\}$.

(P3) *Resolution:* $d(f^*) \leq d(f)$, if f^* is any sharpened version of f, that is: $f^*(x) \leq f(x)$ if $f(x) \leq \frac{1}{2}$ and $f^*(x) \geq f(x)$ if $f(x) \geq \frac{1}{2}$.

(P4) *Symmetry* (about $\frac{1}{2}$) : $d(f^-) = d(f)$.

(P5) *Valuation:* the function d is valuation on $[0,1]^X$, i.e.:

$$d(f \cup g) + d(f \cap g) = d(f) + d(g).$$

(P6) *Generalized additivity:* there exist mappings $s, t : [0,1] \to [0,\infty)$ such that:

$$d(fxg) = d(f) \cdot t(P(g)) + d(g) \cdot s(P(f))$$

for all $f \in [0,1]^X$ and $g \in [0,1]^Y$, X and Y any finite sets.

The properties (P1)-(P4) are natural requirements for a measure of fuzziness. All measures introduced so far satisfy these properties. But the meanings of (P5) and (P6) are not completely clear and therefore their place in such a list is debatable.

2. THE SURVEY OF THE EXISTING MEASURES

We give in this section a brief summary about the various fuzziness measures investigated so far and about their properties.

1. *DeLuca* and *Termini* (1972) [1]

   Measure:

$$d(f) = -K \cdot \sum_{i=1}^{N} \{f(x_i) \cdot \log(f(x_i)) + (1 - f(x_i) \cdot \log(1 - f(x_i))\}$$

   Properties: (P1)-(P5)

2. *Kaufmann* (1975) [6]

   (a) Measure (using the generalized relative Hamming distance):

$$d(f) = \frac{2}{N} \cdot \sum_{i=1}^{N} \mid f(x_i) - f_{\frac{1}{2}}(x_i) \mid$$

   Properties: (P1)-(P5)

   (b) Measure (using the generalized relative Euclidean distance):

$$d(f) = \frac{2}{N^{\frac{1}{2}}} \{\sum_{i=1}^{N} (f(x_i) - f_{\frac{1}{2}}(xi))^2\}^{\frac{1}{2}}$$

   Properties: (P1)-(P5)

3. *Loo* (1977) has proposed a genaral mathematical form for d, thus a large class of measures.

   Measure:

$$d(f) = F\{\sum_{i=1}^{N} c_i \cdot F_i(f(x_i))\},$$

   where $c_i \in R^+$; for all i F_i is a real-valued function such that $F_i(0) = F_i(1) = 0$, $F_i(u) = F_i(1-u)$ for all $u \in [0,1]$, F_i is strictly increasing on $[0, \frac{1}{2}]$; and F is a positive, increasing function.

   Properties: (P1)-(P4) and if F is linear, then (P5) also holds.

   The fuzziness measure introduced by DeLuca and Termini is obtained as special case if $F(x) = x$ and for all i:$c_i = K$ and $F_i(x) = -x \cdot \log(x) - (1-x) \cdot \log(1-x)$.

4. *Trillas* and *Riera* (1978) [2] have proposed another large class of fuzziness measures.
   Measure:
   $$d(f) = \sum_{i=1}^{N} w(x_i) \cdot F(f(x_i)),$$
   where: f is a real-valued function on $[0,1]$, $F(x) = 0$ iff $x \in \{0,1\}$, $F(x)$ is nondecreasing on $[0, \frac{1}{2}]$, $F(x)$ is nonincreasing on $[\frac{1}{2}]$, and w is a positive weight function from X to R.
5. *Emptoz* (1981) [5]
   Measure:
   $$d(f) = \frac{1}{N} \cdot \sum_{i=1}^{N} F(f(x_i)),$$
   where F is a real-valued function on $[0,1]$ such that $F(0) = 0$, $F(x) = F(1-x)$ and F is increasing on $[0, \frac{1}{2}]$.
   Properties: (P1)-(P5)
6. *Ebanks* (1983) [4]
   Measure:
   $$d(f) = \sum_{i=1}^{N} f(x_i) \cdot (1 - f(x_i)), \text{ where } f \in [0,1]^X.$$
   Properties: (P1)-(P6)

3. SOME CRITICAL REMARKS

(1) Various combinations of the desirable properties for the fuzziness measures can be found at the various authors and so it is not possible a unified treatment for this theory.

(2) Related to the first remark there is no point about the axiomatic foundation of the fuzziness measure or a class of measures, because different measures can be derived if we regard various list of properties as axioms. (At the axiomatic foundation is derived the measure from the demanded properties, that is from the axioms.) (See [4], [5].)

(3) The intersection and union operations proposed by Zadeh are not the only possible ones; it is in fact easy to show that one can provide in various ways convenient structures for the class of generalized characteristic functions. Multivalent logics are bases for these set theories. Also the fuzzy set theory arises from the

multivalent logic, but still there is no connection between the fuzziness measure and the fuzzy operators.

(4) There are many applications of the fuzzy set theory but we are not aware of any application of the measure of fuzziness. For this problem a suitable solution is provided by the fuzziness measures induced by continuous conjunctive logical operators. These measures are discussed in the following section.

4. MEASURES INDUCED BY FUZZY OPERATORS

The goals of introducing the multivalent logic and fuzzy set theory were the same, namely to generalize the two-valued logic and the classical set theory, respectively, and to create continuous theories. The multivalent, continuous, conjunctive and disjunctive operators are the mappings $[0,1]x[0,1] \to [0,1]$ with the following properties:

1. commutative: $c(x,y) = c(x,y)$, $d(x,y) = d(x,y)$;
2. associative: $c(c(x,y),z) = c(x,c(y,z))$, $d(d(x,y),z) = d(x,d(y,z))$;
3. monotone: if $y \leq z$, then $c(x,y) \leq c(x,z)$, $d(x,y) \leq d(x,z)$;
4. the correspondence principle is fulfilled: $c(1,x) = x$, $c(0,x) = 0$, $d(0,x) = x$, $d(1,x) = x$;
5. continuous on the interval $[0,1]$;
6. $0 \leq (x,y) \leq \min(x,y)$, $\max(x,y) \leq d(x,y) \leq 1$.

There are some representation theorems for these operators and the above properties arise from the representations of the operators by means of their generator functions [3]. Dombi defined in 1982 fuzziness measure with the help of these operators and their generator functions. But it is possible to generalize the original Dombi's fuzziness measures. Let the fuzziness measure of the fuzzy set f be equal with

$$d(f) = \frac{K}{N} \cdot \sum_{i=1}^{N} c(f(x_i))),$$

where

$c(x,y)$: continuous conjunctive operator,
$n(x)$: continuous negation operator, which is: continuous, strictly monotone decreasing and holds $n(n(x) = x$,
K : normalizing constant, which equals with $(c(q, n(q))$, where q is the neutral value of the negation operator, i.e. $q = n(q)$.

The so defined fuzziness measure can be written in the following form as well:

$$d(f) = \frac{1}{N} \cdot \sum_{i=1}^{N} F(f(x_i)), \text{ where } F(x) = K \cdot c(x, n(x)).$$

and if we use the representation theorems, then

$$F(x) = (1/g^{-1}(2g(q))) \cdot g^{-1} \cdot (g(x) + g(n(x))),$$

where $g(x)$: the generator function of the c(x,y) operator. The fuzziness measures defined above satisfy the properties (P1)-(P5) if the generator function $g(x)$ of the conjunctive operator $c(x,y)$ is strictly convex, differentiable: the negation is concave and $\frac{1}{2}$ in (P1)-(P5) is replaced by q.

This measure can be regarded as the generalization of the measure proposed by Kaufmann using the generalized relative Hamming-distance. This latter measure is a multiple of

$$d(f) = \frac{1}{N}\sum_{i=1}^{N} \mid f(x_i) - f\frac{1}{2}(x_i) \mid .$$

Since Kaufmann used the $\min(x,y)$ operator for the intersection and the $(1-x)$ operator for the pseudo-complement, that is:

$$(f \cap g)(x) = \min\{f(x), g(x)\} \text{ and } f^{-}(x) = 1 - f(x).$$

So we can obtain, that:

$$\mid f(x_i) - f\frac{1}{2}(x_i) \mid = \min\{f(x), \; 1 - f(x)\} = \min\{f(x), f^{-}(x)\} = (f \cap f^{-})(x).$$

This means, that:

$$d(f) = \frac{1}{N} \cdot \sum_{i=1}^{N} \mid f(x_i) - f_{\frac{1}{2}}(x_i) \mid = \frac{1}{N} \cdot \sum_{i=1}^{N} \min\{f(x_i), 1 - f(x_i)\}.$$

If we use general conjunctive $c(x,y)$ negation $n(x)$ operators instead of operators $\min(x,y)$ and $(1-x)$, then we obtain the generalized Dombi's fuzziness measure up to a normalizing constant K.

On the basis of the above construction it is possible to define dual measures with the help of the disjunctive operators. These measures not the fuzziness but the sharpness of a fuzzy set. Let

$$e(f) = \frac{1}{N} \cdot \sum_{i=1}^{N} G(f(x_i)), \text{ where } G(x) = n(F(x)).$$

If the DeMorgan identity holds for three operators, then:

$$G(x) = K \cdot d(x, n(x)) \text{ and } e(f) = d(f) = \frac{K}{N} \cdot \sum_{i=1}^{N} c(f(x_i), n(f(x_i))),$$

where $d(x,y)$ is the continuous disjunctive operator.

In contrast to $d(f)$, $e(f)$ is greater, when f is nearer to the classical characteristic function.

5. APPLICATIONS IN THE FUZZY SET THEORY

Two kinds of problem can arise from the above definition:

1. determination of the adequate measure to a given conjunctive operator:
2. determination of the adequate conjunctive operator to a previously introduced measure.

For the first problem several examples can be given using the available conjunctive operators. For instance in the case of the Hamacher's operator:

$$c(x,y) = (axy)/(1-(1-a)\cdot(x+y-xy)),$$
$$g(x) = -\ln((ax)/(1+(a-l)\cdot x)), \quad \text{and}$$
$$g^{-1}(x) = (e^{-x})/(a+(1-a)\cdot e^{-x}).$$

If we use the $n(x) = 1 - x$ negation operator, then we obtain :

$$f(x) = 4x(1-x) \text{ and } d(f) = \frac{4}{N}\cdot\sum_{i=1}^{N} f(x_i)\cdot(1-f(x_i))$$

for the case $a = 1$.

The second problem is in general more difficult then the first one, because it leads to the solution of a functional equation. Given $F(x)$, we look for the function $g(x)$ with the condition:

$$F(x) = (1/g^{-1}(2g(q)))\cdot g^{-1}(g(x)+g(n(x))).$$

The solution of the second problem does not always exist, because it can happen, that the function obtained from the functional equation does not have the necessary properties to be a generator function.

6. DEDUCTION OF THE FUZZY ENTROPY

In this section we give a derivation of the most famous fuzziness measure, proposed by De Luca and Termini, from the conjunctive operator of the Łukasievics logic by means of a series of operators. The multivalent conjunctive logical operator $c_L(x,y) = \max(0, x+y-1)$ introduced by Łukasievics is associative, continuous, monotone and Archimedean. Therefore on the basis of Ling's theorem [7] there exists a continuous, strictly monotone function g, for which:

$$c_L(x,y) = g^{(-1)}(g(x)+g(y)),$$

where $g^{(-1)}(x)$ is the pseudo-inverse of $g(x)$.
If $g(x)$ is strictly monotone decreasing function, then

$$g^{(-1)}(x) = g^{-1}(x) \qquad \text{if } x \le 1,$$
$$g^{(-1)}(x) = 0 \qquad \text{if } x > 1,$$

and $g(0) = 1$, furthermore $g(x)$ is determined up to a constant multiple factor. This function is called generator function. The generator function of the operator $c_L(x, y)$ is $g(x) = 1 - x$. If we use the Dombi's definition for the fuzziness measure with $n(x) = 1-x$ negation operator, then we obtain an expression of the form "0/0", and so a direct derivation of a fuzziness measure from the Łukasievicz-operator is not possible. It is easy to prove for the Yager's conjunctive operator,

$$c_{Y,a}(x, y) = \max\{0, 1 - ((1-x)^a + (1-y)^a)^{-a}\},$$

that its generator function is $g(x) = (1-x)^a$ and $c_{Y,1}(x, y) = c_L(x, y)$ holds. So it is natural to derive the fuzziness measure belonging to the Łukasievicz-operator as the limit of measures induced by a series of the Yager's operators, that is

$$F_L(f) = \lim_{a \to 1} F_{Y,a}(f).$$

From this we obtain by the L'Hospital-rule, that:

$$F_L(x) = \frac{1}{\ln(2)} \cdot (x \cdot \ln(x) + (1-x) \cdot \ln(1-x).$$

7. APPLICATION IN THE (CONTINUOUS) MULTIVALENT LOGICS

Let us see only the function $F(x)$ without the constant factor instead of the fuzziness measure defined above. The so defined $F(x) = c(x, n(x))$ function can be interpreted as the fuzziness measure of a single element. For instance, if $f \in [0,1]^X$ and $X = \{x_1, x_2, \dots, x_N\}$, then $F(f(x_i)) = c(f(x_i), n(f(x_i)))$ measure the fuzziness of the element x_i by the fuzzy set f. The inequality $c(x, y) \le \min(x, y)$ implies $F(x) \le x$. In the following we shall identify the element x_i of X with its membershipvalue $f(x_i)$, i.e. we shall consider x_i as a multivalent continuous logical variable, whose concrete value is given by a mapping f. So $F(x_i)$ denotes the fuzziness measure of variable x_i.

It is possible to create multivalent logical expressions from variables by means of the conjunctive, disjunctive and negation operators. We want to give a lower and an upper estimation (limit) for the fuzziness of multivalent logical expressions knowing the fuzziness of their variables.

Proposition 1. $c(F(x), F(y)) \leq F(c(x,y))$, $c(F(x), F(y)) \leq F(d(x,y))$ *(lower limit for the fuzziness of conjunction and disjunction).* □

Proof.

$$c(F(x), F(y)) = c(c(x, n(x)), c(y, n(y))) = c(c(x,y), c(n(x), n(y))) \leq$$
$$\leq c(c(x,y), d(n(x), n(y))) = c(c(x,y), n(c(x,y))) = F(c(x,y))$$

is fulfilled on the basis of commutativity, monotonicity, DeMorgan identity and the inequality $c(x,y) \leq d(x,y)$.
This and equality $F(z) = F(n(z))$ imply:

$$c(F(x), F(y)) = c(F(n(x)), F(n(y))) \leq F(c(n(x), n(y))) = F(d(x,y)).$$

■

Corollary . *If $L(x_1, x_2, \ldots, x_n)$ is an arbitrary multivalent logical expression, in which every variable x_i has only one occurrence, then*

$$c(F(x_1), F(x_2), \ldots, F(x_n)) \leq F(L(x_1, x_2, ..., x_n))$$

□

Proposition 2.

If $n(x) \leq y$, *then* $F(d(x,y)) \leq d(F(x), F(y))$.
If $n(x) \geq y$, *then* $F(c(x,y)) \leq d(F(x), F(y))$.

□

Proof.

$$F(d(x,y)) \leq d(F(d(x,y)), F(x)) = d(c(d(x,y), n(d(x,y))), F(x)) =$$
$$= d(F(x), c(d(x,y), c(n(x), n(y)))) \leq d(F(x), c(n(x), n(y))) \leq$$
$$\leq d(f(x), c(y, n(y))) = d(F(x), F(y))$$

is fulfilled on the basis of the properties of the operators and the assumption in the proposition. The consequence of this result is:

$$F(c(x,y)) = F(d(n(x), n(y))) \leq d(F(n(x), F(n(y))) = d(F(x), F(y)),$$

because if $n(x) \geq y$, then $x \leq n(y)$, and so it is possible to use the first inequality. ■

Remark . It is most likely, that Proposition 2 is fulfilled also without its assumptions. This would imply the inequality

$$F(L(x_1, x_2, \ldots, x_n)) \leq d(F(x_1), F(x_2, \ldots, F(x_n))$$

like above.

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