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A NEW APPROACH TO ROUGH SETS AND THEIR RELATION TO FUZZY SETS

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Abstract: Rough sets and fuzzy sets have been often compared to each other. This paper orders this discussion and makes it precise. A new definition of rough sets, more strict and more general, is suggested.

Keywords: Rough set, fuzzy set

1. ORIGINAL DEFINITION OF ROUGH SET

The following definition has been used implicitly or explicitly in [1] and [2], as well as in other papers dealing with rough sets:

Let U be a set called universe, and let R be an equivalence relation on U. The pair $A = (U, R)$ is called an approximation space. Let $[x]_R$ denote for any element $x \in U$ its equivalence class for the relation R. Let $x \subseteq U$ be a subset of U. A rough set R_X corresponding to X is the ordered pair $(\underline{A}(X), \overline{A}(X))$, where $\underline{A}(X)$ and $\overline{A}(X)$ are defined as follows :

$\underline{A}(X) := \{x \in U : [x]_R \subseteq X\}$ (called lower approximation of X)

$\overline{A}(X) := \{x \in U : [x]_R \cap X \neq \emptyset\}$ (called upper approximation of X).

The border $Fr(X)$ of a rough set R_X is defined as $\overline{A}(X) \setminus \underline{A}(X)$.

This definition is illustrated in Fig.1.

2. PREVIOUS DISCUSSION ABOUT THE RELATIONSHIP BETWEEN ROUGH SETS AND FUZZY SETS

The following statements can be found in the literature:

(i) The concept of rough set is wider than the concept fuzzy set ([1]).

(ii) The concept of rough set is wider than subfamily of fuzzy sets: that composed of the membership functions $U \rightarrow \{0, 0.5, 1\}$ ([2]).

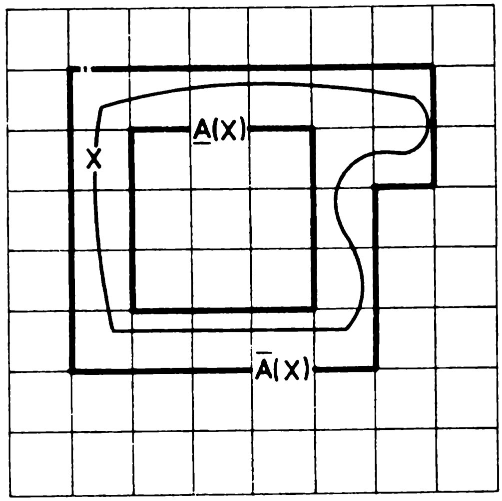

Fig.1

(iii) Both concepts are different.

The last statement has been made without really considering what was wrong with the first two ones. In this paper we will show that the way of reasoning used in [1] and [2] in the discussion about the relationship rough set - fuzzy set was wrong.

We agree with conclusion (iii), but at the same time we will justify that a seemingly reverse statement to (i) holds true:

(iv) The concept of fuzzy set defined in an approximation space is wider than the concept of rough set.

In [1] the question was put, whether it was possible to replace rough sets by fuzzy sets. Then the following definition of a membership function μ_X corresponding to an $X \subseteq U$ was considered:

$$\mu_X(t) := \begin{cases} 1 & t \in \underline{A}(X) \\ 0.5 & t \in Fr(\underline{X}) \\ 0 & t \in U \setminus \overline{A}(X) \end{cases} \quad (1)$$

Subsequently, it was noticed that following inequalities hold in some cases:

$$\mu_{X \cup Y} \neq \max(\mu_X, \mu_Y), \qquad \mu_{X \cup Y} \neq \min(\mu_X, \mu_Y) \tag{2}$$

This can be illustrated with the help of Fig.2. And so, for the elements of the first rectangle $\mu_{X \cup Y}$ will take the value 1 and both μ_X and μ_Y the value 0.5. On the second rectangle $\mu_{X \cup Y}$ equals to 0 and $\min(\mu_X, \mu_Y)$ to 0.5.

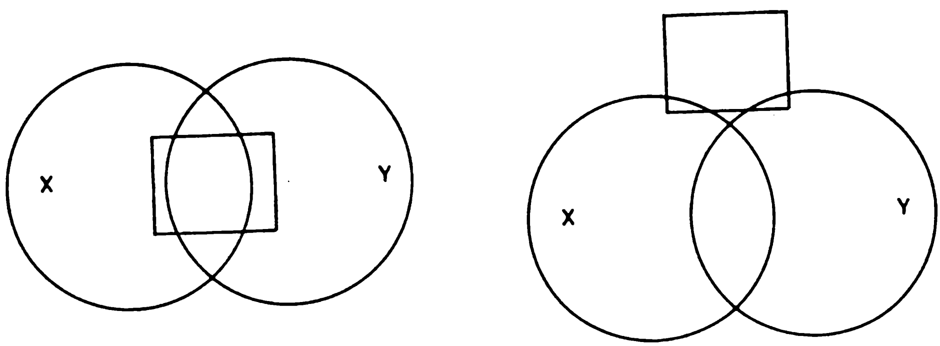

Fig.2.

In some special cases of sets X and Y (e.g. disjoint sets) inequalities in (2) may change to equalities.

The argumentation presented above led in [1] to the conclusion, that in some cases (in those with equalities in (2)) rough sets reduce to fuzzy sets, in other cases do not, hence (1) holds. The same was repeated in [2], with statement (i) corrected to (ii).

3. WEAK POINTS OF THE PREVIOUS DISCUSSION ABOUT THE RELATIONSHIP BETWEEN ROUGH SETS AND FUZZY SETS

Against what could be concluded from the words of statements (i) and (ii), it is not fuzzy sets and rough sets that are compared with each other in [1] and

[2], but it is a certain mapping into the family of fuzzy sets that is compared with rough sets.

Let MF denote this mapping: $MF(X) = \mu_X$, where X is a subset of U and μ_X defined in (1). This mapping has the property of not preserving operations of union and intersection (see (2)), if these operations between $MF(X)$ and $MF(Y)$ (X, Y - any subsets of U) are understood as usual operations on fuzzy sets.

This property of the mapping MF was used as an argument for conclusions (i) and (ii). In our opinion, however, it can not lead to such conclusions, concerning directly fuzzy sets and not mappings into the family of fuzzy sets.

It should be noticed, as well, that rough sets, as defined in section 1, are also a mapping of 2^U, this time into the family of ordered pairs of subsets of U. Let MR denote this mapping: $MR(X) = R_X$, where R_X is defined as in section 1. Hence, in [1] and [2] two mappings MF and MR were compared with each other. All that was stated in the course of this comparison was that MF does not preserve operations of union and intersection. Nothing was said about MR. The reason for that seems to be the fact, that the operations of union and intersection between R_X and R_Y have never been explicitly defined.

In our opinion a comparison between rough sets and fuzzy sets would be possible only if

1) rough sets and fuzzy sets were concepts of the same category, i.e. if there existed a definition of rough sets not corresponding to crisp sets (formally rough set X, as assumed in [1] and [2], is equivalent to the crisp set X);
2) the operations on rough sets were defined.

The following chapter suggests a solution of these problems.

4. A NEW DEFINITION OF ROUGH SET

The following definition of a rough set seems to be more natural:

A rough set RS in the approximation space $A = (U, R)$ is an ordered pair (A, B) such that: $A \subseteq U, B \subseteq U, A \subseteq B$, $A = \cup_{i=1,\ldots,k} E_{n_i}$, $B = \cup_{j=1,\ldots,\ell} E_{n_j}$, where every E_s is any equivalence class of the relation R or the empty set.

The operations on rough sets can be defined as follows:

$$RS_1 \cup RS_2 := (A_1 \cup A_2, B_1 \cup B_2), \qquad RS_1 \cap RS_2 := (A_1 \cap A_2, B_1 \cap B_2),$$

where $RS_1 = (A_1, B_1)$, $RS_2 = (A_2, B_2)$.

5. RELATIONSHIP BETWEEN THE NEW DEFINITION OF ROUGH SETS AND THE OLD ONE

The previously used definition of rough set becomes now a definition of a mapping from the family 2^U into the family of rough sets according to the above definition (the mapping MR). Hence, the new definition does not prevent the various applications of rough sets from being still valid.

With the above new definition of operations on rough sets, the mapping MR has exactly the same properties as the mapping MF as far as the preservation of union and intersection of sets is concerned.

6. COMPARISON BETWEEN ROUGH SETS AND FUZZY SETS WITH THE NEW DEFINITION

The following injective mapping from the family of rough sets in the approximation space A into (and not on) the family of fuzzy sets in A, which preserves the operations defined in section 4, could lead to the conclusion, that fuzzy set is a more general concept than rough set:

Let (A, B) be a rough set. The corresponding fuzzy set has the following membership function:

$$\mu_{(A,B)}(t) = \begin{cases} 1 & t \in A \\ 0.5 & t \in B \setminus A \\ 0 & t \in U \setminus B \end{cases}$$

However, this conclusion would be false, as fuzzy sets normally are not considered in a space equipped with an equivalence relation. That is why only statement iv from section 2 holds true, which concerns not fuzzy sets in general, but fuzzy sets defined in an approximation space equipped with an equivalence relation.

It is exactly this idea of considering a space with an equivalence relation and approximating subsets of this space with equivalence classes, which is the new and most important element in the theory of rough sets. Fuzzy sets are usually used without considering equivalence relations. Hence, the two concepts: that of rough set and that of fuzzy set are different and cannot be really compared with each other.

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