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CONDITIONS FOR OPTIMALITY OF C(S)-VALUED FUNCTIONS

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In this paper an infinite dimensional generalization of Slater optimality, the so-called weak K-optimality is studied. In terms of tangent cones, necessary conditions for weak K-optimality were obtained and applied by Z. Varga in [1] and [2]. Here we prove necessary and sufficient conditions for weak K-optimality of C(S)-valued functions, applying a scalarization method obtained in [3].

1. Preliminary concepts and results

Let Z be a real Banach space and suppose that $K \subset Z$ is a closed, convex and sharp cone. Sharpness means that

$$K \cap (K) = \{0\}.$$

The pair (Z, K) will be called an ordered Banach space.
For any $z_1, z_2 \in Z$ we shall write

$$z_1 \leq z_2 \quad if \quad z_2 - z_1 \in K,$$
$$z_1 \leq z_2 \quad if \quad z_2 - z_1 \in K/\{0\}.$$

In the case of int $K \neq \emptyset$, we shall also use the notation

$$z_1 < z_2 \quad if \quad z_2 - z_1 \in \text{ int } K.$$

Let $A \neq \emptyset$ be an arbitrary set and $f : A \to Z$ a given function.

Definition 1. Suppose that int $K \neq \emptyset$. An element $x_0 \in a$ is said to be weakly K-minimal for the function f if there is no element $x \in A$ such that

$$f(x) < f(x_0).$$

As a synonym of weak K-minimality we shall also say that x_0 is a solution of the weak K-minimization problem

$$(P_w)_{x\in a.}^{f(x)\to \text{weakK-min}}$$

In the particular case of $Z = R^n$ and $K = R^n_{+0}$, for a function $f : A \to R^n$ weak K-minimality is nothing else than Slater minimality.

We shall use the following concept due to Corley /see [4]/.

Definition 2. Given a topological space a, a function $f : A \to Z$ is said to be lower K-closed if for every $z \in Z$ the set $f^{-1}(z - K)$ is closed.

Throughout this paper, let S be a compact Hausdorff topological space and $Z = C(S)$ the Banach space of all real-valued continuous functions on S with the usual supremum norm. Let K be the usual order cone in C(S),

$$K = \{z \in C(S) \mid z(s) \geq 0 \quad (s \in S)\}.$$

Consider an arbitrary function $\rho \in C(S)$ with $\rho(s) > 0 \quad (s \in S)$, or equivalently $\rho \in \text{int}K$.

Definition 3. We define the norm generated by the weight ρ in the following way

$$\| z \|_\rho = \| \rho z \| = \max_{s\in S} \rho(s) \mid z(s) \mid \quad (z \in C(S)).$$

It is clear that C(S) is a Banach space with respect to this norm.

Definition 4. The function f is called lower K-bounded with a lower K-bound $\varphi \in C(S)$ if

$$\varphi < f(x) \quad (x \in A),$$

or equivalently

$$R_f \subset \varphi + \text{int}K.$$

We assume that the function f is lower K-bounded with a lower K-bound φ, and for each $\varphi \in \text{int}K$, and consider the scalar minimization problem

$$(P_\rho) \quad \begin{array}{l} \| f(x) - \varphi \|_\rho \to \min \\ x \in A. \end{array}$$

Now we recall two theorems from [3] which relate vector optimization problem (P_w) to the family (P_ρ) of scalar optimization problems.

Theorem 1. Suppose that an element $x_0 \in A$ is a solution of problem (P_w). Then there exists a function $\rho \in \text{int}K$ such that x_0 is a solution of problem (P_ρ).

Theorem 2. Let A be a compact topological space and $f : A \to C(S)$ lower K-closed. Suppose that $x_0 \in A$ is a solution of the problem (P_ρ) for some $\rho \in \text{int}K$. Then x_0 is weakly K-minimal for f.

We also need some basic concepts and results from non smooth optimization theory /see [5]/.

Definition 5. Given a subset $B \subset Z$ and a point $z_0 \in B$, the normal cone of B at z_0 is defined in the following way

$$N(z_0) = \partial \delta_B(z_0)$$

where $\delta_B : Z \to \bar{R}$ denotes the indication function of B:

$$\delta_B(z) = \begin{cases} +\infty & \text{for} \quad z \in Z/B \\ 0 & \text{for} \quad z \in B. \end{cases}$$

Given a function $g : Z \to R$, consider the following problem

$$(P_B) \quad \begin{array}{l} g(z) \to \min \\ z \in B. \end{array}$$

From [6] we recall

Proposition 1. Let $g : Z \to R$ be a convex function, $B \subset Z$ a convex set. Assume that g is continuous at some point of B and $z_0 \in B$. Then z_0 is a solution of the problem (P_B) if and only if

$$0 \in \partial g(z_0) + N_B(z_0).$$

2. A necessary condition for optimality

Theorem 3. Let f be lower K-bounded with a lower K-bound $\varphi \in C(S)$ and suppose that $R_f \subset C(S)$ is convex. If $x_0 \in A$ is a solution of problem (P_w) then there exists a regular Borel measure μ in S such that

$$1^0 \quad \int_S f(x_0)d\mu = \min_{x\in A} \int_S f(x)d\mu,$$

$$2^0 \quad \int_S [\varphi - f(x_0)]d\mu = \min_{s\in S} \rho(s)[\varphi(s) - f(x_0)(s)],$$

$$3^0 \quad |\mu| \leq \max_{s\in S} \rho(s)$$

where $|\mu|$ is the total variation of the measure μ.

Proof. 1^0 Let x_0 be a solution of (P_w). Then by theorem 1 there exists a weight $\rho \in \text{int}K$ such that x_0 is a solution of the problem

$$(P_\rho) \quad \begin{array}{l} \| f(x) - \varphi \|_\rho \rightarrow \min \\ x \in A. \end{array}$$

Hence $f(x_0)$ is a solution of the problem

$$(Q_\rho) \quad \begin{array}{l} g(z) := \min \| z - \varphi \|_\rho \rightarrow \min \\ z \in R_f \end{array}$$

By proposition 1 we have:

$$0 \in \partial g(f(x_0)) + N_{R_f}(f(x_0)).$$

This means that there exists a functional $z^* \in C_\rho(S)^*$ such that

$$(1) \qquad z^* \in \partial g(f(x_0))$$

where $C_\rho(S)$ denotes the space C(S) with the norm $\| \ \|_\rho$, and

$$(2) \qquad -z^* \in N_{R_f}(f(x_0)).$$

From the definition of the normal cone it is easy to see that

$$< z^* m f(x_0) - z > \leq 0 \qquad (z \in R_f).$$

In other words

(3) $$< z^*, f(x_0) - f(x) > \leq 0 \qquad (x \in A).$$

Moreover, for all $z \in C(S)$ we have

(4) $$|< z^*, z >| \leq \| z^* \|_\rho \| z \|_\rho =$$

(5) $$\| z^* \|_\rho \| \rho z \| \leq \| z^* \|_\rho \| \rho \| \| z \|$$

where $\| z^* \|_\rho$ denotes the norm of z^* as an element of $C_\rho(S)^*$.

From (4) and (5) we obtain that $z^* \in C(S)^*$ and

(6) $$\| z^* \| \leq \| z^* \|_\rho \| \rho \|$$

Therefore by the Riesz representation theorem there exists a regular Borel measure μ in S such that

$$< z^*, z > = \int_S z d\mu \qquad (z \in C(S)).$$

This inequality (3) implies that

$$\int_S f(x_0) d\mu \leq \int_S f(x) d\mu \qquad (x \in A)$$

which is the same as 1^0.

2^0 From the relation (1), for all $z \in C(S)$ we have

$$< z^* m z m f(x_0) > \leq g(z) - g(f(x_0))$$

(7) $$= \| z - \varphi \|_\rho - \| f(x_0) - \rho \|_\rho \leq \| z - f(x_0) \|_\rho .$$

Let $u \in C(S)$ arbitrary and define

(8) $$z = f(x_0) + u.$$

Substituting (8) into (7) we get

(9)
$$< z^*, u > \leq \| u \|_\rho \quad (u \in C(S)).$$

Hence
(10)
$$\| z^* \|_\rho \leq 1.$$

Taking $z := \varphi$ in (7) we have

$$< z^*, \varphi - f(x_0) > \leq - \| f(x_0) - \varphi \|_\rho \tag{11}$$

or

$$< z^*, f(x_0) - \varphi > \leq \| f(x_0) - \varphi \|_\rho \,. \tag{12}$$

On the other hand, with $u = f(x_0) - \varphi$, (9) implies

$$< z^*, f(x_0) - \varphi > = \| f(x_0) - \varphi \|_\rho \,. \tag{13}$$

In terms of the measure μ, obtained in the proof of 1^0 from (1) we obtain

$$\int_S [f(x_0) - \varphi] d\mu = \max_{s \in S} \rho(s) \mid f(x_0)(s) - \varphi(s) \mid =$$
$$\max_{s \in S} \rho(s)[f(x_0)(s) - \varphi(s)]$$

Therefore

$$\int_S [\varphi - f(x_0)] d\mu = \max \rho(s)[f(x_0)(s) - \varphi(s)] =$$
$$\min_{s \in S} \rho(s)[\varphi(s) - f(x_0)(s)]$$

which is 2^0.

3/ Inequalities (6) and (10) imply that

$$\mid \mu \mid = \| z^* \| \leq \| z^* \|_\rho \| \rho \| \leq \| \rho \|$$

Hence 3^0 holds. Theorem 3 is proved.

Under additional topological assumptions, we have the following sufficient condition:

Theorem 4. Let A be a compact topological space $f : A \to C(S)$ lower K-closed and lower K-bounded with a lower K-bound $\varphi \in C(S)$. Suppose that for some $x_0 \in A$ there exists a regular Borel measure μ in S and a weight $\rho \in \mathrm{int}K$ such that relations $1^0, 2^0$ and 3^0 hold.

Then x_0 is weakly K-minimal for f.

Proof. We clearly have

$$\begin{aligned}\| f(x_0) - \varphi \|_\rho &= \max_{s\in S} \rho(s) \mid f(x_0)(s) - \varphi(s) \mid = \\ &= \max_{s\in S} \rho(s)[f(x_0)(s) - \varphi(s)] = \\ &= -\min_{s\in S} \rho(s)[\varphi(s) - f(x_0)(s)].\end{aligned}$$

Let $x \in A$ be arbitrary. Then by 1^0 and 2^0 we obtain

$$\begin{aligned}&\| f(x_0) - \varphi \|_\rho = \int_S [f(x_0) - \varphi] d\mu = \\ &= \int_S f(x_0) d\mu - \int_S \varphi d\mu \le \int_S f(x) d\mu - \int_S \varphi d\mu = \\ &= \int_S [f(x) - \varphi] d\mu \le \| \mu \| \| f(x) - \varphi \|\end{aligned}$$

Since f is lower K-bounded with a lower K-bound φ, by condition 3 we get

$$\begin{aligned}&\| \mu \| \| f(x) - \varphi \| = \| \| \mu \| [f(x) - \varphi] \| = \\ &= \max_{s\in S} \{ \mid \mu \mid [f(x)(s) - \varphi(s)] \} \le \\ &\le \max_{s\in S} \rho(s)[f(x)(s) - \varphi(s)] = \\ &= \| f(x) - \varphi \|_\rho .\end{aligned}$$

Therefore

$$\| f(x_0) - \varphi \|_\rho \le \| f(x) - \varphi \|_\rho \quad (x \in A)$$

i.e. x_0 is a solution of the problem

$$(P_\rho) \quad \begin{array}{l} \| f(x) - \varphi \|_\rho \to \min \\ x \in A. \end{array}$$

Hence, by theorem 2, x_0 is weakly K-minimal for f. Theorem 4 is proved.

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