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FFT METHOD FOR BIORTHOGONAL EXPANSION WITH RESPECT THE INTEGRATED WALSH FUNCTIONS

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1. Introduction

In this work we investigate expansions of continuous functions with respect to the integrated Walsh system. We give the biorthogonal system of the integrated Walsh system in explicit form and prove that a subsequence of partial sums of the biorthogonal expansion of a continuous functions is uniformly convegent. It is also proved that in general, this is not true for the whole sequence. In fact we prove that there exists a continuous function such that the whole sequence diverges at a point. A subsequence of partial sums is represented in terms of the Walsh transform of an appropriate function. It is proved that the Fast Walsh Transformer /FWT/ can be used for computing very efficiently the coefficients of the expansion and the partial sums for $n = 2^N, n \in \mathcal{N}$.

2. Biorthogonal expansion of functions with respect to the respect to the integrated Walsh system

Let $X = \{f \in C[0,1],\ f(0) = f(1)\}$. The dual space X' of X is the space of functions of bounded variation and every linear functional on X can be written in the form

$$< f, g > := \int_0^1 f dg \qquad (f \in X, g \in x').$$

It is easy to check that the integrated Walsh system

$$H_n(x) := \int_0^x w_n(t)dt \qquad (x \in [0,1], n \in \mathcal{N})$$

and the modified Walsh-system

$$\tilde{w}_n = -W_n \qquad (n \in \mathcal{N})$$

$$\tilde{w}_0 = \begin{cases} -1 & (0 \le x < 1) \\ 0 & x = 1 \end{cases}$$

are biorthogonal, i.e.

$$< J_m, w_n >= \delta_{mn} \qquad (m, n \in \mathcal{N}).$$

Moreover, $J = (J_n, n \in \mathcal{N})$ is a closed system in X.

The biorthogonal expansion of $f \in X$ with respect J is defined by

$$f v \sum_{k=0}^{\infty} (\int_0^1 f d\tilde{w}_k) J_k.$$

The partial sums of this expansion are denoted by

$$S_m f := \sum_{n=0}^{m-1} (\int_0^1 f d\tilde{w}_k) J_k \quad (m = 1, 2, ...).$$

It is convenient to use the following modulus of continuity

$$w(f, 2^{-n}) =$$

$$= \sup\{| f(x) - f(y) | \ x, y \in [\frac{k}{2^n}, \frac{k+1}{2^n}], \ \ k = 0, 1, ..., 2^n - 1\}.$$

For the subsequence $(S_{2^n} f, n \in \mathcal{N})$ the following result is true.

Theorem 1. Let $f \in X$ and $n \in \mathcal{N}$. Then partial sum $S_{2^n} f$ is an intepolating polygonal of f with nodes $\frac{k}{2^n}$, $k = 0, l, ..., 2$ i.e.

$$(S_{2^n} f_{(\frac{k}{2^n})}) = f(\frac{k}{2^n}) \text{ for } k = 0, 1, ..., 2^n$$

and $(S_{2^n} f)$ converges uniformly to f on $[0, 1]$ as $n \to \infty$. Moreover

$$\| f - S_{2^n} f \|_{\infty} \le w(f, 2^{-n})$$

where $w(f, \delta)$ $(\delta > 0)$ is the modulus of continuity.

Proof. To prove the theorem we fix a function $f \in X$ with $f(0) = f(1) = 0$, and $n \in \mathcal{N}$. For any $x \in [0, 1]$ the partial sums can be written in the form

$$(S_m f)_{(x)} = \sum_{k=0}^{m-1}\Big(\int_0^1 f(t)d_t w_k(t)\Big)J_k(x)$$
$$= \int_0^1 f(t)d_t\Big(\sum_{k=0}^{m-1} \tilde{w}_k(t)J_k(x)\Big)$$

where d_t means that the integral is taken with respect to t.

Let us define

$$M_m(x,t) := \sum_{k=0}^{m-1} \tilde{w}_k(t)J_k(x).$$

Then

$$(S_m f)_{(x)} = \int_0^1 f(t)d_t M_m(x,t).$$

Now

$$M_m(x,t) = \int_0^x \sum_{k=0}^{m-1} \tilde{w}_k(t)w_k(u)du$$
$$= \begin{cases} -\int_0^x \sum_{k=0}^{m-1} w_k(t+u)du & 0 \le t < 1 \\ -\int_0^x \Big(\sum_{k=0}^{m-1} w_k(t+u) - 1\Big)du & t = 1 \end{cases}$$
$$= \begin{cases} -\int_0^x D_m(t+u)du & 0 \le t < 1 \\ x - \int_0^x D_m(t+u)du & t = 1 \end{cases}$$

where $D_m(u) = \sum_{k=0}^{m-1} w_k(u)$ is the Walsh-Dirichlet kernel. In the case $m = 2^n$ the kernel M_m has a simple form. Namely using Paley's lemma for any $[x \in \frac{k}{2^n}, \frac{k+1}{2^n}), k = 0, 1, ..., 2^n - 1$ we have

$$M_{2^n}(x,t) = \begin{cases} -1 & \text{if} \quad 0 \le t < \frac{k}{2^n} \\ -2^n(x - \frac{k}{2^n}) & \text{if} \quad \frac{k}{2^n} \le t \le \frac{k+1}{2^n} \\ 0 & \text{if} \quad \frac{k+1}{2^n} \le t < 1 \\ x & \text{if} \quad t = 1. \end{cases}$$

In particular, for $x = \frac{k}{2^n}$, we have

$$M_{2^n}(\frac{2}{2^n}, t) = \begin{cases} -1 & \text{if} \quad 0 \le t < \frac{k}{2^n} \\ 0 & \text{if} \quad \frac{k}{2^n} \le t < 1 \\ \frac{k}{2^n} & \text{if} \quad t = 1. \end{cases}$$

and consequently

$$(S_{2^n} f)_{(\frac{k}{2^n})} = \int_0^1 f d_t M_{2^n}(\frac{k}{2^n}, t) = f(\frac{k}{2^n})$$

for $k = 1, ..., 2^n$. In the case $k = 0$ obviously

$$f(0) = (S_{2^n} f)_{(0)} = 0.$$

Fix $n \in \mathcal{N}$ and $x \in [0,1)$. Then there exists a k such that $0 \le k < 2^n$ and $n \in I(k,n) := [\frac{k}{2^n}, \frac{k+1}{2^n})$. Since $(S_{2^n} f)$ interpolates the function f at $[\frac{k}{2^n}, \frac{k+1}{2^n})$, there exist $\alpha, \beta \in [0,1]$ with $\alpha + \beta = 1$ such that

$$(S_{2^n} f)_{(x)} = \alpha (S_{2^n} f)_{(\frac{k}{2^n})} + \beta (S_{2^n} f)_{(\frac{k+1}{2^n})}.$$

Consequently

$$\begin{aligned} \mid (S_{2^N} f_{(x)} - f(x) \mid &\le \alpha \mid (S_{2^n} f)_{(\frac{k}{2^n})} - f(x) \mid + \beta \mid (S_{2^n} f)_{(\frac{k+1}{2^n})} f(x) \mid \\ &= \alpha \mid f(\frac{k}{2^n}) - f(x) \mid + \beta \mid f(\frac{k+1}{2^n}) - f(x) \mid \\ &\le w(f, I(k,n)) \end{aligned}$$

where $w(f, I(k,n)) = \sup_{x,y \in I[k,n)} \mid f(x) - f(y) \mid$. Therefore

$$\| S_{2^n} f - f \|_\infty \le (f, 2^{-n}).$$

3. Fourier analysis and syntesis

For any $f \in X$ we can represent the partial sums $S_{2^n} f, n \in \mathcal{N}$ with respect to the integrated Walsh system using the Walsh transform of a suitable function. In fact we have the following

Theorem 2. Let $f \in X$ and $N \in \mathcal{N}$. Then partial sum

(1) $\quad (S_{2^n} f) := \sum_{k=0}^{2^N-1} (\int_0^1 f d\tilde{w}_k) J_k$

can be written as

(2) $\quad (S_{2^n} f_{(x)} = \int_0^x \sum_{k=0}^{2^N-1} F_N(k) w_k(t) dt$

where $F_N(k), k = 0, 1, ..., 2^N - 1$ are the Walsh-Fourier coefficients of the function $F_N := [0,1] \to \mathcal{R}$ defined in the following way

$$F_N(x) = 2^N (f(\frac{k+1}{2^N} - f(\frac{k}{2^N})), \frac{k}{2^N} \le x < \frac{k+1}{2^N}, k = 0, 1, ..., 2^N - 1.$$

Proof. Denote by $w_k(a-0)$ the left-hand side limit of the function w_k at the point a. Then using the Abel transform, the Stieljes integral of $f \in X$ with respect to $w_k (0 \le k < 2^N)$ can be written in the form

$$\begin{aligned}\int_0^1 f d\tilde{w}_k &= \sum_{j=1}^{2^n-1} f(\frac{j}{2^N})(w_k(\frac{j}{2^N}) - w_k(\frac{j}{2^N} - 0)) \\ &= - \sum_{j=1}^{2^n-1} f(\frac{j}{2^N})(w_k(\frac{j}{2^N}) - w_k(\frac{j-1}{2^N})) \\ &= \sum_{j=0}^{2^n-1} (w_k(\frac{j}{2^N})(f(\frac{j+1}{2^N} - f(\frac{k}{2^N})).\end{aligned}$$

By the definition of F_N we get

$$\begin{aligned}\int_0^1 f d\tilde{w}_k &= 2^{-N} \sum_{j=0}^{2^N-1} F_N(\frac{j}{2^N}) w_k(\frac{j}{2^N}) = \\ &= \int_0^1 F_N(t) w_k(t) dt = F_N(k).\end{aligned}$$

Consequently

$$S_{2^N} f = \sum_{k=0}^{2^N-1} \left(\int_0^1 f d\tilde{w}_k\right) J_k$$
$$= \sum_{k=0}^{2^N-1} F_N(k) J_k = \int_0^x \sum_{k=0}^{2^N-1} F_N(k) w_k,$$

the theorem is proved.

When we represent the partial sums S_{2^N} in the form given in the above theorem, we can use the algorithm called Fast-Walsh transform given by F. Schipp and P. Simon in [2]. The latter makes it possible to compute the Walsh transform F_N very efficiently.

Using this algorithm we require $N2^N$ arithmetic operations to get the values $F_N(k), k = 0, 1, ..., 2^N - 1$.

Identity (2) can be used to compute the partial sums $(S_{2^N} f)_{(x)}$ at the points $x = \frac{k}{2^N}(k = 0, ..., 2^N)$, i.e. make Fourier systems. To this end we first use FWT to compute the sums

$$\sum_{k=0}^{2^N-1} F_N(f) w_k(t) \quad (t = \frac{\ell}{2^N}, \ell = 0, ..., 2^N)$$

and then compute the integral of this function.

4. Divergence of the biorthogonal expansion of a continuous function

We know that the subsequence $\{S_{2^N} f\}$ converges uniformly to f for any $f \in X$. In what follows, we shall prove that there exist a function $f_0 \in X$ and a point $x_0 \in [0, 1]$ such that the whole sequence $\{(S_n f_0)_{(x_0)}\}$ diverges.

Let $f_n(t) = \text{sign} D_{m_n}(t) J_{2^{2n}}(t) 2^{2n}, n \in \mathcal{N}$ where

$m_n = \sum_{k=0}^{n-1} 2^{2k} \quad (n \in \mathcal{N}).$

We define

$$\Delta_n f_n = S_{m_{n+1}} f_n - S_{2^{2n}} f_n.$$

Since sign $D_{m_n} J_{2^{2n}}$ is absolutely continuous, integration by parts gives

$$(\Delta_n f_n)_{(x)} = \sum_{k=2^{2n}}^{2^{2n}+m_n-1} (\int_0^1 f_n(t)d\tilde{w}_k)J_k(x) =$$

$$= \sum_{k=2^{2n}}^{2^{2n}+m_n-1} (\int_0^1 \operatorname{sign} D_{m_n}(t)J_{2^n}(t)2^{2n}d\tilde{w}_k(t))J_k(x) =$$

$$= 2^{2n} \sum_{k=2^{2n}}^{2^{2n}+m_n-1} (\int_0^1 \operatorname{sign} D_{m_n}(t)w_{2^{2n}}(t)w_k(t)dt)J_k(x) =$$

$$= 2^{2n} \int_0^x \int_0^1 \operatorname{sign} D_{m_n}(t)w_{2^{2n}}(t) \sum_{k=2^{2n}}^{2^{2n}+m_n-1} w_k(t+s)dtds =$$

$$= 2^{2n} \int_0^x \int_0^1 \operatorname{sign} D_{m_n}(t)w_{2^{2n}}(s) \sum_{k=0}^{m_n-1} w_k(t+s)dtds.$$

Now for $x = \frac{1}{2^{2n+1}}$ we have that $w_k(s) = 1$ for $0 \le k \le 2^{2n+1}$. If $0 \le s \le x = \frac{1}{2^{2n+1}}$, then

$$(\Delta_n f_n)_{(\frac{1}{2^{2n+1}})} = 2^{2n} \int_0^{\frac{1}{2^{2n+1}}} \int_0^1 \operatorname{sign} D_{m_n}(t) \sum_{k=0}^{m_n-1} w_k(t)dtds.$$

It is known /see [2]/ that

$$\int_0^1 | D_{m_n}(t) | dt \ge \frac{n}{4} \quad (n \in \mathcal{N}),$$

consequently

$$(\Delta_n f_n)_{(\frac{1}{2^{2n+1}})} = 2^{2n} \int_0^{\frac{1}{2^{2n+1}}} \int_0^1 | D_{m_n}(t) | dtds$$

$$= 2^{2n} \frac{1}{2^{2n+1}} \int_0^1 | D_{m_n}(t) | dt \ge \frac{n}{8}.$$

Since $\| f_n \| = 1$, we have

$$\| \Delta_n f_n \| \geq \frac{n}{8}.$$

Therefore, using the fact that $\sup_n \| S_{2^{2n}} \| < \infty$, we get

$$\| S_{m_{m+1}} \| \geq \| \Delta_n f_n \| - \| S_{2^{2n}} \| \to \infty \text{ as n} \to \infty.$$

Then by the Banach-Steinhaus theorem, there exist a function $f_0 \in X$ and $x_0 \in [0,1]$ such that sequence $\{(S_n f_0)_{(x_0)}\}$ diverges.

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