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ON THE NUMERICAL SOLVING OF NONLINEAR VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS

A.A. BOJELDAIN

Abstract. In this paper existence and uniqueness theorems, for first order, second order, and m-th order Nonlinear Volterra Integro-Differential Equations, abbreviated (NVIDE), are presented in a Banach space using the contraction mapping principle.

Introduction

Since, in general, when applying a numerical method to a VIDE, linear or nonlinear, one usually begins by assuming that the problem has a solution, we introduce here some existence and uniqueness theorems to settle this problem (note that these Theorems are also valid for linear VIDE).

At first the existence and uniqueness theorem for the NVIDE $x'(t,p) = f(t,p,x(t,p)) + \int_{t_0}^{t} K(t,s,p,x(s,p))ds$ with the initial condition $x(t_0,p) = a(p)$ is proved in a Banach space equipped with the Bielecki's type norm [3] given by the weighted norm $\| x \| := \max_{t,p} exp(r(t)) \mid x(t,p) \mid$; p is an arbitrary finite parameter, and $r(t,p)$ will be explained in the Theorems.

In Theorem 2 we discuss a NVIDE on the general form $x'(t) = f(t, x(t), IKx)$, where $IKx := \int_{t_0}^{t} K(t,s,x(s))ds$, and f depends also nonlinearly on IKx, with $x(t_0) = a$, while Theorem 3 is for the case when this NVIDE has a finite parameter p. Thm. 4 and THm. 5 are devoted to the second order NVIDE $x''(t) = f(t,x(t),x'(t),IKx)$, having the initial condition $x(t_0) = a$ & $x'(t_0) = b$; where $IKx := \int_{t_0}^{t} K(t,s,x(s),x'(s))ds$, and for the parametrized case respectively.

A generalization for the m-th order NVIDE and for the parametrized one:

$$x^{(m)}(t,p) = f(t,p,x(t,p),x'(t,p),........,x^{(m-1)}(t,p))$$

are proved in Thm. 6 and Thm. 7 respectively, where

$$IKx := \int_{t_0}^{t} K(t,s,x(s,p),x'(s,p),\ldots\ldots,x^{(m-1)}(s,p))ds.$$

(Note that each Theorem is proved in a Banach space equipped with a max weighted norm i.e. a Bielecki's norm [3]).

1. Parametrized first order NVIDE - additive case

Theorem 1.

The parametrized NVIDE

$$x'(t,p) = f(t,p,x(t,p)) + \int_{t_0}^{t} K(t,s,p,s(s,p))ds \tag{1.1}$$

with the initial condition (i.c.) $x(t_0,p) = a(p)$; where K is functionally dependent, and p is a finite parameter, has a unique solution $x(t,p) \in C'[t_0,T]$ in the Banach space $D \subset B$ given by

$$D : t_0 \leq t \leq T,\ t_0 \leq s \leq T,\ \mid x - a \mid \leq b; \tag{1.2}$$

if the following conditions are satisfied:

1: $f(t,p,x(t,p))$ is continuous for every t and x in D.

2: f satisfies a Lipschitz condition on x in D with a Lipschitz coefficient $\ell_1(p)$ i.e.

$$\mid f(t,p,x(t,p)) - f(t,p,y(t,p)) \mid \leq \ell_1(p) \mid x - y \mid \tag{1.3}$$

for every $x(t,p)$ and $y(t,p)$ in D.

3: $K(t,s,p,x(s,p))$ is continuous for all t,s,x in D.

4: K satisfies a Lipschitz condition on x in D with a Lipschitz coefficient $\ell_2(p)$ i.e.:

$$\mid K(t,s,p,x(s,p)) - K(t,s,p,y(s,p)) \mid \leq \ell_2(p) \mid x - y \mid \tag{1.4}$$

for every $x(s,p)$ and $y(s,p)$ in D, and B is equipped with

$$(1.5)\qquad \| x \| := \max_{t,p} exp(-r(t)) \mid x(t,p) \mid$$

i.e. the Bielecki's norm [3], where $r(t) := cL(t-t_0)$ for an integer $c > 1$, and $L := \max_p(\ell_1(p), \ell_2(p), 1)$.

Proof.

It is easy to show that (1.1) is equivalent to the integral equation

$$(1.6)\qquad \begin{aligned} x(t,p) = a(p) + \int_{t_0}^{t} f(s,p,s(s,p))ds + \\ + \int_{t_0}^{t}\int_{t_0}^{s} K(t,u,p,x(u,p))duds. \end{aligned}$$

In order to have a fixed point problem, we choose the r.h.s. of (1.6) to be our nonlinear operator $Q(x)$ and consider the difference $\mid Q(x) - Q(y) \mid$.

$$(1.7)\qquad \begin{aligned} \mid Q(x) - Q(y) \mid \leq \int_{t_0}^{t} \mid f(s,p,x(s,p)) - f(s,p,y(s,p)) \mid ds + \\ + \int_{t_0}^{t}\int_{t_0}^{s} \mid K(t,u,p,x(u,p)) - K(t,u,p,y(u,p)) \mid duds. \end{aligned}$$

Make use of (1.3) and (1.4) in (1.7) to obtain

$$(1.8)\qquad \mid Q(x) - Q(y) \mid \leq \ell_1(p) \int_{t_0}^{t} \mid x - y \mid ds + \ell_2(p) \int_{t_0}^{t}\int_{t_0}^{s} \mid x - y \mid duds$$

$$(1.9)\qquad \mid Q(x) - Q(y) \mid \leq L \int_{t_0}^{t} \mid x - y \mid ds + L \int_{t_0}^{t}\int_{t_0}^{s} \mid x - y \mid duds.$$

Multiply the r.h.s. of (1.9) by $exp(-r(t))\cdot exp(r(t))$ - i.e. we multiply it by 1 - thus (1.9) becomes

$$\mid Q(x)-Q(y)\mid\leq L\int_{t_0}^{t} exp(-r(s))\mid x-y\mid exp(r(s))ds+$$

$$+L\int_{t_0}^{t}\int_{t_0}^{s} exp(-r(u))\mid x-y\mid exp(r(u))duds\leq$$

(1.10)

$$\leq L\int_{t_0}^{t}\max_{s,p}(exp(-r(s))\mid x-y\mid)exp(r(s))ds+$$

$$+L\int_{t_0}^{t}\int_{t_0}^{s}\max_{up}(exp(-r(u))\mid x-y\mid)exp(r(u))duds.$$

According to (1.5) we have

$$\mid Q(x)-Q(y)\mid\leq\parallel x-y\parallel(L\int_{t_0}^{t} exp(r(s))ds+$$

(1.11)

$$+L\int_{t_0}^{t}\int_{t_0}^{s} exp(r(u))duds).$$

Perform the integrals in the r.h.s. of (1.11) to get

$$\mid Q(x)-Q(y)\mid\leq\parallel x-y\parallel(\frac{1}{c}+\frac{1}{c^2})[exp(r(t))-1]- \tag{1.12}$$

$$-\frac{t-t_0}{c}\leq\parallel x-y\parallel(\frac{1}{c}+\frac{1}{c^2})[exp(r(t)-1];$$

since $(t-t_0)\leq T-t_0=:z$. Now multiply both sides of the inequality (1.12) by $exp(-r(t))$, this gives

$$exp(-r(t))\mid Q(x)-Q(y)\mid\leq\parallel x-y\parallel(\frac{1}{c}+\frac{1}{c^2})[1-exp(-r(t))]\leq \tag{1.13}$$

$$\leq \| x - y \| \left(\frac{1}{c} + \frac{1}{c^2}\right)(1 - exp(-czL)).$$

The inequality (1.13) is true for all $t \in [t_0, T]$ and thus for the maximum of its l.h.s. as well, since $Q(x)$ is continuous over D - which is given by (1.2) -, and the most r.h.s. of (1.13) is independent of t. Therefore

$$\max_{t,p}(exp(-cL(t - t_0)) \mid Q(x) - Q(y) \mid) \leq \tag{1.14}$$

$$\leq \left(\frac{1}{c} + \frac{1}{c^2}\right)(1 - exp(-cLz)) \| x - y \| .$$

According to (1.5) we obtain the following inequality

$$\| Q(x) - Q(y) \| \leq \left(\frac{1}{c} + \frac{1}{c^2}\right)(1 - exp(-cLz)) \| x - y \| . \tag{1.15}$$

Since $(1 - exp(-cLz)) < 1$ for every finite positive $c, L,$ and z, so $c = 2$ guarantees that

$$0 < q := \frac{3}{4}(1 - exp(-2Lz)) < 1 \tag{1.16}$$

for every finite parameter $p, L \geq 1$, and $z > 0$.

Hence $Q(x)$ is a contraction operator, and thus the classical Banach's fixed point theorem is now applicable. □

2. First order (NVIDE) in the general form.

Theorem 2.

Consider the NVIDE having the general form

$$x'(t) = f(t, x(t), IKx), \quad x(t_0) = a; \tag{2.1}$$

where f depends also nonlinearly on IKx; such that

$$IKx := \int_{t_0}^{t} K(t, s, x(s))ds. \tag{2.2}$$

If the conditions

1. $K(t,s,x(s))$ is continuous for every t, s in $[t_0, T]$, and satisfies a Lipschitz condition on x in D i.e.

$$(2.3) \qquad | K(t,s,x(s)) - K(t,s,y(s)) | \leq \ell_1 | x - y |$$

for every $x(s)$ and $y(s)$ in D which is given by

$$(2.4) \qquad D : t_0 \leq t \leq T,\ t_0 \leq s \leq T,\ | x - a | \leq b.$$

2. $f(t,x(t),IKx)$ is continuous for t in $[t_0,T]$, and Lipschitzian on x and IKx in D i.e.

$$(2.5) \qquad | f(t,x(t),IKx) - f(t,y(t),IKy) | \leq \ell_2(| x - y | + \ell_1 I | x - y |)$$

such that the following notation is used:

$$(2.6) \qquad | IKx - IKy | \leq I | Kx - Ky | \leq I\ell_1 | x - y | =: \ell_1 I | x - y |$$

for every x, y in D, and B is equipped with the norm

$$(2.7) \qquad \| x \| := \max_t exp(-r(t)) | x(t) |;$$

for $r(t) := cL(t - t_0)$, $L := max(\ell_1, \ell_2, 1)$ and c(integer)> 1.
If they are satisfied then (2.1) has a unique solution $x(t) \in C'[t_0, T]$ in D.

Proof.

Similarly, as mentioned before, (2.1) is equivalent to the integral equation

$$(2.8) \qquad x(t) = a + \int_{t_0}^{t} f(s, x(s), IKx) ds.$$

In order to have a fixed point problem, we choose the r.h.s. of (2.8) to be our nonlinear operator $Q(x)$ and consider the difference

$$|Q(x)-Q(y)| \leq \int_{t_0}^{t} |f(s,x(s),IKx)-f(s,y(s),IKy)|\,ds. \tag{2.9}$$

Make use of (2.3) and (2.5) in (2.9) to obtain

$$|Q(x)-Q(y)| \leq \ell_2 \int_{t_0}^{t} (|x-y|+\ell_1 I|x-y|)ds, \tag{2.10}$$

$$|Q(x)-Q(y)| \leq L \int_{t_0}^{t} (|x-y|+LI|x-y|)ds. \tag{2.11}$$

Multiply the r.h.s. of (2.11) by $exp(-r(t))exp(r(t))$ to get

$$\begin{aligned}|Q(x)-Q(y)| \leq L \int_{t_0}^{t} (&exp(-r(s))|x-y|exp(r(s))+\\ &+LIexp(-r(s))|x-y|exp(r(s)))ds\end{aligned} \tag{2.12}$$

then take the max of the r.h.s. i.e.

$$\begin{aligned}|Q(x)-Q(y)| \leq L \int_{t_0}^{t} [&\max_s(exp(-r(s))|x-y|)exp(r(s))+\\ &+\max_s(exp(-r(s))|x-y|)exp(r(s))]ds\end{aligned} \tag{2.13}$$

and according to (2.7) this becomes

$$|Q(x)-Q(y)| \leq L\|x-y\| \int_{t_0}^{t} [exp(r(s))+LIexp(r(s))]ds. \tag{2.14}$$

Note that from (2.2), (2.6), and the definition of the auxiliary function $exp\ (r(t))$, we have:

$$LIexp(r(s))ds = L\int_{t_0}^{t} exp(cL(s-t_0))ds = \tag{2.15}$$

$$= \frac{1}{c}[exp(cL(t-t_0))-1].$$

Replace this in (2.14), then perform its r.h.s. integration to obtain

$$\begin{aligned} | Q(x)-Q(y) | \leq \| x-y \| [\frac{1}{c}(exp(cL(t-t_0))-1)+ \\ +\frac{1}{c^2}(exp(cL(t-t_0))-1)-\frac{t-t_0}{c}]; \end{aligned} \tag{2.16}$$

however, $t-t_0 \leq T-t_0 =: z$, thus

$$\begin{aligned} | Q(x)-Q(y) | \leq \| x-y \| [\frac{1}{c}(exp(cL(t-t_0))-1)+ \\ +\frac{1}{c^2}(exp(cL(t-t_0))-1)] = \frac{c+1}{c^2}[exp(cL(t-t_0))-1] \| x-y \| . \end{aligned} \tag{2.17}$$

Multiply both sides of (2.17) by $exp(-r(t))$; hence

$$\begin{aligned} exp(-cL(t-t_0)) \, | Q(x)-Q(y) | \leq \\ \leq \frac{c+1}{c^2}[1-exp(-cL(t-t_0))] \| x-y \| \leq \\ \leq \frac{c+1}{c^2}[1-exp(-cLz)] \| x-y \| . \end{aligned} \tag{2.18}$$

Similar reasoning to that used in obtaining (1.14) in page (6) leads to:

$$\begin{aligned} \max_t(exp(-cL(t-t_0)) \, | Q(x)-Q(y) |) \leq \\ \leq \frac{c+1}{c^2}[1-exp(-cLz)] \| x-y \|; \end{aligned} \tag{2.19}$$

which, according to our norm definition, becomes

$$\| Q(x) - Q(y) \| \leq \frac{c+1}{c^2}[1 - exp(-cLz)] \| x - y \| . \tag{2.20}$$

It is obvious that $c \doteq 2$ makes the coefficient

$$\frac{c+1}{c^2}[1 - exp(-cLz)] = \frac{3}{4}[1 - exp(-2Lz)] =: q, \tag{2.21}$$

and thus $0 < q < 1$ for every finite, $L \geq 1, z > 0$.

Therefore $Q(x)$ is a contraction operator and Banach's Fixed Point Theorem is applicable.□

3. First order NVIDE with parameter

Theorem 3.

If the NVIDE (2.1) is parametrized i.e.

$$x'(t,p) = f(t, x(t,p), IKx), \quad x(t_0,p) = a(p); \tag{3.1}$$

where f depends also nonlinearly on IKx, such that

$$IKx := \int_{t_0}^{t} K(t,s,p,x(s,p))ds, \tag{3.2}$$

and p is an arbitrary finite parameter.

Let us pose the following conditions:

1. $K(t,s,p,x(s,p))$ is continuous for every t, s in $[t_0, T]$ and satisfies the Lipschitz condition

$$| K(t,s,p,x(s,p)) - K(t,s,p,y(s,p)) | \leq \ell_1(p) \mid x - y \mid \tag{3.3}$$

for every $x(s,p)$ and $y(x,p)$ in $D \subset B$ which is given by

$$D : t_0 \leq t \leq T,\ t_0 \leq s < T,\ \mid x - a \mid \leq b. \tag{3.4}$$

2. $f(t,p,x(t,p),IKx)$ is continuous for t in $[t_0,T]$, and Lipschitzian on x and IKx in D i.e.

$$\begin{aligned} &\mid f(t,p,x(t,p),IKx) - f(t,p,y(t,p),IKy) \mid \leq \\ &\leq \ell_2(p)(\mid x - y \mid + \ell_1(p)I \mid x - y \mid) \end{aligned} \tag{3.5}$$

where the following notation is used

$$\mid IKx - IKy \mid \leq I \mid Kx - Ky \mid \leq I\ell_1(p) \mid x - y \mid = \ell_1(p)I \mid x - y \mid \tag{3.6}$$

for every x, y in D; here B is equipped with the norm

$$\| x \| := \max_{t,p} exp(-r(t)) \mid x(t,p) \mid; \tag{3.7}$$

i.e. the Bielecki's norm [3]; where $r(t) := cL(t - t_0)$ for an integer $c > 1$, and $L := max(\ell_1(p), \ell_2(p), 1)$.

If the conditions 1 and 2 are satisfied, then (3.1) has a unique solution $x(t,p) \in C'[t_0,T]$ in D which is given by (3.4).

Proof.

Similarly, as in the previous Thm. (3.1) is equivalent to the integral equation

$$x(t,p) = a(p) + \int_{t_0}^{t} f(s,p,x(s),IKx)ds. \tag{3.8}$$

The proof now follows exactly the same steps that used in the proof of the previous Theorem, except that the max; in this case is considered w.r.t. t and p.□

4. Second order NVIDE in the general form

Theorem 4.

If the 2-nd order NVIDE in the form

$$x''(t) = f(t, x(t), x'(t), IKx) \tag{4.1}$$

has the initial conditions $x(t_0) = a_0$ and $x'(t_0) = a_1$; similarly f depends also nonlinearly on IKx, where

$$IKx := \int_{t_0}^{t} K(t, s, x(s), x'(s))ds, \tag{4.2}$$

satisfies the conditions.

1. $K(t, s, x(s), x'(s))$ is continuous for every t and s in $[t_0, T]$ and satisfies the Lipschitz condition;

$$\begin{aligned} &| K(t, s, x(s), x'(s)) - K(t, s, y(s), y'(s)) | \leq \\ &\leq \ell_1(| x - y | + | x' - y' |) \end{aligned} \tag{4.3}$$

in $D \subset B$ which is given by

$$D : t_0 \leq t \leq T,\ t_0 \leq s \leq T,\ | x - a_0 | \leq b_0,\ | x' - a_1 | < b_1 \tag{4.4}$$

2. $f(t, x(t), x'(t), IKx)$ is continuous for t in $[t_0, T]$, and Lipschitzian on x, x', and IKx in D i.e.

$$\begin{aligned} &| f(t, x(t), x'(t), IKx) - f(t, y(t), y'(t), IKy) | \leq \\ &\leq \ell_2(| x - y | + | x' - y' + \ell_1 I(| x - y | + | x' - y' |)) \end{aligned} \tag{4.5}$$

such that the following notation is used (see (3.6)):

$$(4.6) \qquad | IKx - IKy | \leq \ell_1 I(| x - y | + | x' - y' |);$$

where B is equipped with the following norm

$$(4.7) \qquad \| x \| := \max_t [exp(-r(t))(| x(t) | + | x'(t) |)],$$

$r(t) := cL(t - t_0)$, c(integer)> 1, and $L := max(\ell_1, \ell_2, 1)$; then (5.1) has a unique solution $x(t) \in c''[t_0, T]$ in D.

Proof.

Integrating both sides of (5.1) twice from t_0 to t leads to the equivalent integral equation

$$(4.8) \qquad x(t) = a_0 + a_1 \int_{t_0}^{t} ds + \int_{t_0}^{t} \int_{t_0}^{s} f(u, x(u), x'(u), IKx) du ds.$$

In order to have a fixed point problem, we choose the r.h.s. of (4.8) to be our nonlinear operator $Q(x)$; say and consider the difference $| Q(x) - Q(y) |$.

$$(4.9) \qquad | Q(x) - Q(y) | \leq$$

$$\leq \int_{t_0}^{t} \int_{t_0}^{s} | f(u, x(u), x'(u), IKx) - f(u, y(u), y'(u)), IKx) | du ds.$$

Make use of (4.3) and (4.5) in (4.9); then multiply the r.h.s. by $exp(-r(t))$ $\cdot exp(r(t))$ to get the following inequality (after considering the max at the r.h.s.)

$$(4.10) \qquad | Q(x) - Q(y) | \leq$$

$$\leq L \int_{t_0}^{t} \int_{t_0}^{s} [\max_u (exp(-r(u))(| x - y | + | x' - y' |)) exp(r(u)) +$$

$$+LI\max_u(exp(-r(u))(\mid x-y\mid+\mid x'-y'\mid))exp(r(u))]duds,$$

and according to (4.7) this becomes

$$\mid Q(x)-Q(y)\mid\leq \tag{4.11}$$

$$\leq L\parallel x-y\parallel\int_{t_0}^{t}\int_{t_0}^{s}[exp(r(u))+LIexp(r(u))]duds$$

Similarly from (4.2), (4.6), and the definition of the auxiliary function, $exp(r(t))$, we have:

$$LIexp(r(u))du=L\int_{t_0}^{t}exp(cL(u-t_0)du= \tag{4.12}$$

$$=\frac{1}{c}[exp(cL(t-t_0))-1].$$

Replace this in (4.11), then perform its r.h.s. double integral and put $(t-t_0)\leq T-t_0=:z$ to obtain

$$\mid Q(x)-Q(y)\mid\leq\parallel x-y\parallel[\frac{1}{c^2L}(exp(cL(t-t_0))-1)+$$

$$+\frac{1}{c^3L}(exp(cL(t-t_0))-1)-\frac{2cz+2z+cLz^2}{2c^2}]< \tag{4.13}$$

$$<\frac{c+1}{c^3}[exp(cL(t-t_0))-1]\parallel x-y\parallel,$$

since $(1/c^3L)<1/c$ and $(1/c_2L)<(1/c^2)$.

Now multiply both sides of (4.13) by $exp(-r(t))$ and mimic the same steps from inequality (2.18) up to the end of the Proof of Thm. 2 with

$$\parallel Q(x)-Q(y)\parallel\leq\frac{3}{8}[1-exp(-2Lz)]\parallel x-y\parallel, \tag{4.14}$$

thus $Q(x)$ is a contraction operator, since for every finite $L\geq 1$ and $z>0; 0<q:=(3/8)[1-exp(-2Lz)]<1.\square$

5. Second order NVIDE with parameter

Theorem 5.

The parametrized 2-nd order NVIDE

$$x''(t,p) = f(t,p,x(t,p),x'(t,p),IKx) \tag{5.1}$$

with the i.c. $x(t_0,p)a_0(p)$, $x'(t_0,p) = a_1(p)$; where f depends also nonlinearly on IKx such that

$$IKx := \int_{t_0}^{t} K(t,s,p,x(s,p),x'(s,p))ds \tag{5.2}$$

has a unique solution $x(t,p) \in C''[t_0,T]$ in $D \subset B$; where

$$D: t_0 \le t \le T,\ t_0 \le s \le T,\ \mid x - a_0 \mid \le b_0,\ \mid x' - a_1 \mid \le b_1 \tag{5.3}$$

and the Banach space B is equipped with the norm

$$\| x \| := \max_{t,p}[exp(-r(t))(\mid x(t,p) \mid + \mid x'(t,p) \mid)]; \tag{5.4}$$

if it i.e. (5.1) satisfies the following conditions

1. $K(t,s,p,x(s,p),x'(s,p))$ is continuous for every t, s in $[t_0,T]$, and satisfies the Lipschitz condition;

$$\mid K(t,s,p,x(s,p),x'(s,p)) - K(t,s,p,y(s,p),y'(s,p)) \mid \le \tag{5.5}$$
$$\le \ell_1(p)(\mid x - y \mid + \mid x' - y' \mid)$$

for every $x(s,p)$, $x'(s,p)$, $y(s,p)$, and $y'(s,p)$ in D.

2. $f(t,p,x(t,p),x'(t,p),IKx)$ is continuous for t in $[t_0,T]$, and Lipschitzian on x, x' and IKx in D i.e.

$$(5.6) \qquad | f(t,p,x(t,p),x'(t,p),IKx) - f(t,p,y(t,p),y'(t,p),IKy) | \leq$$
$$\leq \ell_2(p)(| x - y | + | x' - y' | + \ell_1(p)I(| x - y | + | x' - y' |))$$

such that the following notation is used (see (3.6)):

$$(5.7) \qquad | IKx - IKy | \leq \ell_1(p)I(| x - y | + | x' - y') |$$

The proof of this Thm. follows exactly the same steps that used in proving the previous Theorem but the max must be considered w.r.t. t as well as p.□

6. m-th order NVIDE without parameter.

Theorem 6.

Consider the following NVIDE of order m

$$(6.1) \qquad x^{(m)}(t) = f(t, x(t), x'(t), \ldots\ldots, x^{(m-1)}(t), IKx)$$

with the i.c. $x(t_0) = a_0, x'(t_0) = a_1, \ldots\ldots, x^{(m-1)}(t_0) = a_{m-1}$ - here also f depends nonlinearly on IKx; where

$$(6.2) \qquad IKx := \int_{t_0}^{t} K(t, s, x(s), x'(s), \ldots\ldots, x^{(m-1)}(s))ds$$

and pose the following conditions

1. $K(t, s, x(s), x'(s), x''(s), \ldots\ldots, x^{(m-1)}(s))$ is continuous for every t, s in $[t_0, T]$ and satisfies the Lipschitz condition:

$$| K(t, s, x(s), x'(s), \ldots\ldots, x^{(m-1)}(s)) -$$
$$(6.3) \qquad -K(t, s, y(s), y'(s), \ldots\ldots, y^{(m-1)}(s)) | \leq$$
$$\leq \ell_1(| x - y | + | x' - y' | + \ldots\ldots + | x^{(m-1)} - y^{(m-1)} |)$$

in the $(m+2)$-dimensional region $D \subset B$ given by

$$D : t_0 \leq t \leq T,\ t_0 \leq s \leq T,\ | x^{(i)} - a_i | < b_i, \tag{6.4}$$

for $i = 0, 1, 2, \ldots\ldots, m-1$; where $x^{(0)} = x$.

2. $f(t, x(t), x'(t), x''(t), \ldots\ldots, x^{(m-1)}(t))$ is continuous for every t in $[t_0, T]$, and satisfies the following Lipschitz condition in D

$$\begin{aligned} &| f(t, x(t), x'(t), \ldots\ldots, x^{(m-1)}(t)), IKx)- \\ &-f(t, y(t), y'(t), \ldots\ldots, y^{(m-1)}(t)), IKy) | \leq \\ &\leq \ell_2[| x - y | + | x' - y' | + \ldots\ldots + |^{(m-1)} - y^{(m-1)} | + \\ &+\ell_1 I(| x - y | + | x' - y' | + \ldots\ldots + | x^{(m-1)} - y^{(m-1)} |)] \end{aligned} \tag{6.5}$$

such that the following notation is used (see (3.6)):

$$| IKx - IKy | \leq \ell_1 I(| x - y | + | x' - y' + \ldots\ldots + | x^{(m-1)} - y^{(m-1)} |). \tag{6.6}$$

If (6.1) satisfies the conditions 1 and 2 then it has a unique solution $x(t) \in C^m[t_0, T]$ in D; where B is equipped with the norm

$$\| x \| := \max_t (exp(-r(t)) \sum_{i=0}^{m-1} | x^{(i)}(t) |); \tag{6.7}$$

where $x^{(0)}(t) = x(t)$, and $r(t) := cL(t - t_0)$ for an integer $c > 1$ and $L := max(\ell_1, \ell_2, 1)$.

Proof.

By integrating both sides of (6.1) step by step m times from t_0 to t one can obtain the following equivalent integral equation

$$x(t) := a_0 + a_1 \int_{t_0}^{t} ds_1 + a_2 \int_{t_0}^{t} \int_{t_0}^{s} ds_2 ds_1 +$$

$$(6.8)\qquad +a_3\int_{t_0}^{t}\int_{t_0}^{s_1}\int_{t_0}^{s_2} ds_3ds_2ds_1+.....+a_{m-1}\int_{t_0}^{t}\int_{t_0}^{s_1}....\int_{t_0}^{s_{m-2}} ds_{m-1}....ds_1+$$

$$+\int_{t_0}^{t}\int_{t_0}^{s_1}\int_{t_0}^{s_2}....\int_{t_0}^{s_{m-1}} f(s_{m-1}x(s_m),x'(s_m),....,x^{(m-1)}(s_m),IKx)ds_m....ds_1.$$

(m times)

Choose the r.h.s. of (6.8) to be the nonlinear operator $Q(x)$, then consider the difference

$$(6.9)\qquad \mid Q(x)-Q(y)\mid\leq$$

$$\leq\int_{t_0}^{t}\int_{t_0}^{s_1}\int_{t_0}^{s_2}.....\int_{t_0}^{s_{m-1}} \mid f(s_{m-1},x(s_m),x'(s_m),.....,x^{(m-1)}(s_m),IKx)-$$

$$-f(s_{m-1},y(s_m),y'(s_m),.....,y^{(m-1)}(s_m),IKy)\mid ds_m....ds_1.$$

Using (6.3) and (6.5) in this inequality implies that

$$(6.10)\qquad \mid Q(x)-Q(y)\mid\leq$$

$$\leq\ell_2\int_{t_0}^{t}\int_{t_0}^{s_1}\int_{t_0}^{s_2}....\int_{t_0}^{s_{m-1}} [\mid x-y\mid+\mid x'-y'\mid+......+\mid x^{(m-1)}-y^{(m-1)}\mid+$$

$$+\ell_1I(\mid x-y\mid+\mid x'-y'\mid+.....+\mid x^{(m-1)}-y^{(m-1)}\mid)]ds_m....ds_1.$$

Now multiply the r.h.s. by $exp(-r(t))\cdot exp(r(t))$, then take the max after using $L:=max(\ell_1,\ell_2,1)$ to get:

$$(6.11)\qquad \mid Q(x)-Q(y)\mid\leq$$

$$\leq\int_{t_0}^{t}\int_{t_0}^{t}\int_{t_0}^{s_1}\int_{t_0}^{s_2}....\int_{t_0}^{s_{m-1}} [\max_{s_m}(exp(-r(s_m))\sum_{i=0}^{m-1}\mid x^{(i)}-y^{(i)}\mid)exp(r(s_m))+$$

$$+LI\max_{s_m}(exp(-r(s_m))\sum_{i=0}^{m-1}\mid x^{(i)}-y^{(i)}\mid)exp(r(s_m))]ds_m....ds_1$$

(6.12)
$$\mid Q(x)-Q(y)\mid\leq\parallel x-y\parallel\cdot$$

$$\cdot L[\int_{t_0}^{t}\int_{t_0}^{s_1}\int_{t_0}^{s_2}....\int_{t_0}^{s_{m-1}}[exp(r(s_m))+LIexp(r(s_m))]ds_m....ds_1].$$

Now replace $LIexp(r(s_m))$ by $(1/c)(exp(r(s_m))-1)$, (see (4.12)), in (6.12) and perform the integration of its r.h.s. to end with

$$\mid Q(x)-Q(y)\mid\leq\parallel x-y\parallel\cdot$$

$$L[(\frac{1}{(cL)^m}+\frac{1}{c^{m+1}L^m})(exp(cL(t-t_0))-1)-\sum_{i=1}^{m-1}\frac{(t-t_0)^i}{(i)!(cL)^{m-i}}-$$

(6.13)
$$-\sum_{i=1}^{m-1}\frac{(t-t_0)^i}{(i)!(c^{m+1-i})L^{m-1}}-\frac{(t-t_0)^m}{m!c}]\leq\parallel x-y\parallel\cdot$$

$$L[(\frac{1}{(cL)^m}+\frac{1}{c^{m+1}L^m})(exp(cL(t-t_0))-1)-\sum_{i=1}^{m-1}\frac{(z)^i}{(i)!(cL)^{m-i}}-$$

$$-\sum_{i=1}^{m-1}\frac{(z)^i}{(i)!(c^{m+1-i}L^{m-1}}-\frac{(z)^m}{m!c}]$$

and this inequality can be majorized as follows

(6.14)
$$\mid Q(x)-Q(y)\mid\leq\parallel x-y\parallel\cdot$$

$$\cdot L[(\frac{1}{(cL)^m}+\frac{1}{c^{m+1}L^m})(exp(cL(t-t_0))-1)].$$

Similarly multiply both sides of (6.14) by $exp(-r(t))$ and mimic the same steps of the proof of Thm.2 from inequality (2.18) up to the end to obtain:

$$\parallel Q(x)-Q(y)\parallel\leq\parallel x-y\parallel\cdot$$

$$(6.15) \qquad \cdot L[(\frac{1}{(cL)^m} + \frac{1}{c^{m+1}L^m})(1 - exp(-cLz))] \leq$$

$$\leq (\frac{1}{c^m} + \frac{1}{c^{m+1}})(1 - exp(-cLz)) \parallel x - y \parallel;$$

since $(1/c^{m+1}L^{m-1}) \leq (1/c^{m+1})$, and $(1/c^m L^{m-1}) < (1/c^m)$.

It is clear that $c = 2$ is enough to make; $0 < q := ((2+1)/2^{m+1})(1 - exp(-2Lz)) < 1$ for every finite $L \geq 1$ and $z > 0$, whence $Q(x)$ is a contraction operator and thus the Banach's fixed point Theorem is applicable.□

7. m-th order NVIDE with parameter.

Theorem 7.

Consider the parametrized m-th order NVIDE:

$$(7.1) \qquad x^{(m)}(t,p) = f(t,p,x(t,p),x'(t,p),....,x^{(m-1)}(t,p),IKx)$$

having the i.c. $x(t_0,p) = a_0(p),\ x'(t_0,p) = a_1(p),....,\ x^{(m-1)}(t_0,p) = a_{m-1}(p)$, where p is an arbitrary finite parameter; here also f depends nonlinearly on

$$(7.2) \qquad IKx := \int_{t_0}^{t} K(t,s,p,x(s,p)x'(s,p),....,x^{(m-1)}(s,p))ds.$$

Let the following conditions be posed

1. $K(t,s,p,x(s,p),x'(s,p),x''(s,p),....,x^{(m-1)}(s,p))$ is continuous for every t,s in $[t_0,T]$ and satisfies the Lipschitz condition:

$$| K(t,s,p,x(s,p),x'(s,p),....,x^{(m-1)}(s,p)) -$$

$$(7.3) \qquad -K(t,s,p,y(s,p),y'(s,p),....,y^{(m-1)}(s,p)) | \leq$$

$$\leq \ell_1(p)(| x-y | + | x'-y' | + ... + | x^{(m-1)} - y^{(m-1)} |)$$

in the $(m+2)$-dimensional region $D \subset B$ given by

$$(7.4) \qquad D : t_0 \leq t \leq T,\ t_0 \leq s \leq T,\ | x^{(i)} - a_i | < b_i$$

where $i = 0, 1, 2, ..., m-1, x^{(0)} = x$, and p is finite.

2. $f(t, p, x(t,p), x'(t,p), x''(t,p), ..., x^{(m-1)}(t,p))$ is continuous for every t in $[t_0, T]$, and satisfies the following Lipschitz condition:

$$\begin{aligned} (7.5) \qquad & | f(t,p,x(t,p),x'(t,p),....,s^{(m-1)}(t,p)), IKx) - \\ & - f(t,p,y(t,p),y'(t,p),....,y^{(m-1)}(t,p)), IKy) | \leq \\ & \leq \ell_2(p)[| x - y | + | x' - y' | +....+ | x^{(m-1)} - y^{(m-1)} | + \\ & + \ell_1(p) I(| x - y | + | x' - y' | +....+ | x^{(m-1)} - y^{(m-1)} |)] \end{aligned}$$

for every $x^{(i)}(t,p)$ and $y^{(i)}(t,p)$ in D; $i = 0, 1, ..., m-1$; such that the following notation is used (see (3.6)):

$$(7.6) \quad | IKx - IKy | \leq \ell_1(p) I(| x - y | + | x' - y' | +....+ | x^{(m-1)} - y^{(m-1)} |)$$

If (7.1) satisfies the conditions 1 and 2 then it has a unique solution $x(t,p) \in C^m[t_0, T]$ in D, where B is equipped with the norm

$$(6.7) \qquad \| x \| := \max_{t,p}(exp(-r(t)) \sum_{i=0}^{m-1} | x^{(i)}(t,p) |);$$

where $x^{(0)}(t,p)$, and $r(t) := cL(t - t_0)$ for an integer $c > 1$ and $L := \max_p(\ell_1(p), \ell_2(p), 1)$.

Proof.

The proof follows exactly, in the same manner, as the proof of the previous Theorem step by step but the maximum is considered w.r.t. t and p.□

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