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ITERATIONS CONVERGING FASTER THAN NEWTON'S METHOD TO THE SOLUTIONS OF NONLINEAR EQUATIONS IN BANACH SPACE

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Introduction.

Consider the equation

$$F(x) = 0 \tag{1}$$

where F is nonlinear operator mapping a subset E of a normed space X into a normed space Z. We assume that F is k-times Frechet-differentiable on E [4], [8]. Suppose that an approximation x_n to a solution x^* of equation (1) by the equation

$$F(x_n) + F'(x_n)(x - x_n) + \frac{1}{2}F''(x_n)(x - x_n)^2 + \ldots + \frac{1}{k!}F^{(k)}(x_n)(x - x_n)^k = 0, \tag{2}$$

where $F^{(j)}(x_n), j = 1, 2, \ldots, k$ are j-linear operators corresponding to the j-th Frechet-derivative of F at x_n [7], [8], $n = 0, 1, 2 \ldots$.

For fixed $x_n, z \epsilon E, n = 0, 1, 2, \ldots$, define the linear operators on E by

$$L_{n,k}(z)(x) = F'(x_n)(x) + \frac{1}{2}F''(x_n)(z - x_n)(x) + \ldots + \frac{1}{k!}F^{(k)}(x_n)(z - x_n)^{k-1}(x), \quad n = 0, 1, 2, \ldots. \tag{3}$$

Using (3), (2) can equivalently be written as

$$F(x_n) + L_{n,k}(x)(x - x_n) = 0, \qquad n = 0, 1, 2, \ldots. \tag{4}$$

Moreover, if we assume that the linear operators $L_{n,k}(x)$ are invertible on E, (4) becomes

$$x = T_{n,k}(x) \tag{5}$$

where $T_{n,k}$ are nonlinear operators defined on E by

$$T_{n,k}(x) = x_n - L_{n,k}(x)^{-1}F(x_n), \qquad n = 0,1,2,\dots. \tag{6}$$

Equation (5) suggests that the approximation x_{n+1} can be found implicitly using the iteration

$$X_{n+1} = T_{n,k}(x_{n+1}), \qquad n = 0,1,2,\dots. \tag{7}$$

Note that for $k = 1$ the above iteration becomes explicit and reduces to the Newton-Kantorovich iteration for solving (1) [4], [6].

Assuming that the linear operator $L_{n,1}(x_0)$ has a bounded inverse on some $D \subset E$, the Newton-Kantorovich theorem ensures that if

$$a = a(x_0) = 2bl \parallel L_{n,1}(x_0)^{-1}F(x_0) \parallel \le 1, \tag{8}$$

$$r = r(x_0) = \frac{1}{bl}(1 - \sqrt{1-a}), \tag{9}$$

where l is the Lipschitz constant of $L_{n,1}$ on $D \subset E$ and

$$b = b(x_0) = \parallel L_{n,1}(x_0)^{-1} \parallel .$$

Then equation (1) has a solution

$$x^* \epsilon \bar{B}(x_0, r) = \{x \epsilon X \mid \parallel x - x_0 \parallel < r\} \subset D$$

which is a unique solution of (1) in the open ball $B(x_0, \bar{r})$ with radius

$$\bar{r} = \bar{r}(x_0) = \frac{1}{bl}(1 + \sqrt{1-a}). \tag{10}$$

Moreover, iteration (7), for $k = 1$ converges to x^* quadratically. That is,

$$\parallel x_{n+1} - x^* \parallel = 0(\parallel x_n - x^* \parallel^2), \quad n = 0,1,2,\dots. \tag{11}$$

Suppose that there exists $x^* \epsilon E$ which is obtained as the limit of the iteration (7) as $n \to \infty$ and k is fixed. Then by (7)

$$F(x^*) = F(\lim_{n\to\infty} x_{n+1}) = 0, \tag{12}$$

that is x^*, so obtained is a solution of the equation (1).

Here we provide sufficient conditions for the convergence of iteration (7) to a solution x^* of equation (1). We also show that if F is $(k+1)$-times Frechet-differentiable, then the following estimate holds

$$\| x_{n+1} - x^* \| = 0 \ (\| x_n - x^* \|^{k+1}), \quad n = 0, 1, 2, \ldots . \tag{13}$$

The above result improves (11) for $k > 1$. However, iteration (7) becomes implicit. More precisely, (5) becomes a polynomial equation of degree k on E. Polynomial equations have already been studied in [1], [2], [6], [7] and the references there. In fact, the continued fraction technique [3], the contraction mapping principle with perturbations [1], [5] and the Newton- Kantorovich theorem [4], [6] are some of the techniques that have been applied for the solution of (5).

Due to the particular properties of polynomials, equation (5) is, in general, easier to handle than equation (1), especially on finite dimensional spaces. There are problems where the desired error tolerance $\varepsilon > 0$ is such that the number of iterations required by (7) for $k = 1$ due to (11) is very large. It is in those cases where the solution of (5) will reduce the number of iterations required to achieve the same accuracy ε due to (13).

The evaluation of the iterate x_{n+1} in (7) will itself require an iteration of the form

$$x_{n+1,m+1} = T_{n,k}(x_{n+1,m}), \quad m = 0, 1, 2, \ldots \tag{14}$$

for fixed n and some initial guess $x_{n+1,0}$.

Because of rounding or discretization error in the evaluation of $T_{n,k}$, an approximate sequence $z_{n+1,m}$ is produced in place of the exact sequence $x_{n+1,m}$. That is

$$z_{n+1,m+1} = \tilde{T}_{n,k}(z_{n+1,m}), \quad m = 0, 1, 2, \ldots, \tag{15}$$

where the $\tilde{T}_{n,k}$ are related with the $T_{n,k}$, $\quad n = 0, 1, 2, \ldots$.

In [4, 12.2.1] it was proved that if the operators $T_{n,k}$ are all contractions on some closed set $D_1 \subset E$ and $\{z_{n+1,m}\} \subset D_1$, then for $x_{n+1,0} \epsilon D_0$ with $T_{n,k}(D_0) \subset D_0 \subset D_1$ iteration (14) converges to a unique fixed point x^*_{n+1} of $T_{n,k}$ in D_0.

Moreover,

$$\lim_{n \to \infty} z_{n+1,m} = x^*_{n+1}$$

$$\text{if and only if } \lim_{m \to \infty} \| T_{n,k}(z_{n+1,m}) - z_{n+1,m+1} \| = 0 \tag{16}$$

To illustrate the procedures described above a simple example is provided when $X = C$, the set of complex numbers.

I. Main results.

From now on we assume that $X = Y$ is a Banach space and state the main result.

Theorem. Let $F : E \subset X \to X$ be a nonlinear operator which is $(k+1)$-times Frechet-differentiable on E. Assume that the linear operators $L_{n,k}(x_{n+1})$ are invertible with bounded inverse on some closed ball $B^* \subset E$ such that $\{x_n\} \subset B^*$, $n = 0, 1, 2, \ldots$.

Set,

$$\| L_{n,k}(x_{n+1})^{-1} \| \leq c_n \leq c, \tag{17}$$

and

$$\frac{1}{(k+1)!} \max_{\tilde{x} \epsilon B^*} \| F^{(k+1)}(\tilde{x}) \| \leq d_n \leq d, \quad n = 0, 1, 2 \ldots \tag{18}$$

for some $c, c_n, d, d_n > 0$ guaranteed to exist by the hypotheses on $L_{n,k}(x_{n+1})$, F and the standard estimate (given in [4] for example) for (18).

Then if

$$0 < cd < 1 \tag{19}$$

the following are true:

(i) the iteration $\{x_n\}$ given by (7) converges to a solution $x^* \epsilon B^*$ of equation (1);

(ii) moreover

$$\| x_{n+1} - x^* \| = 0(\| x_n - x^* \|^{k+1}), \quad n = 0, 1, 2, \ldots.$$

Proof. We have by (7)

$$\| x_{n+1} - x_n \| = \| L_{n,k}(x_{n+1})^{-1} F(x_n) \|$$

$$\begin{aligned}
&\leq c \parallel F(x_n) - F(x_{n-1}) - F'(x_{n-1})(x_n - x_{n-1}) - \\
&- \ldots - \frac{1}{k!} F^{(k)}(x_{n-1})(x_n - x_{n-1})^k \parallel \\
&\leq c \frac{1}{(k+1)!} \max_{\tilde{x} \epsilon B^*} \parallel F^{(k+1)}(\tilde{x}) \parallel \cdot \parallel x_n - x_{n-1} \parallel^{k+1} \\
&\leq cd \parallel x_n - x_{n-1} \parallel^{k+1} \\
&\leq (cd)(cd)^{k+1} \parallel x_{n-1} - x_{n-2} \parallel \\
&\ldots \\
&\leq (cd)^{(k+1)n} \parallel x_1 - x_0 \parallel .
\end{aligned} \tag{20}$$

Also, $p = 2, 3 \ldots$

$$\parallel x_{n+p} - x_n \parallel \leq \parallel x_{n+p} - x_{n+(p-1)} \parallel + \parallel x_{n+(p-1)} - x_n \parallel . \tag{21}$$

Now,

$$\begin{aligned}
\parallel x_{n+p} - x_{n+(p-1)} \parallel &\leq (cd) \parallel x_{n+(p-1)} - x_{n+(p-2)} \parallel^{k+1} \\
&\ldots \\
&\leq (cd)^{(k+1)(p-1)+1} \parallel x_{n+1} - x_n \parallel^{k+1},
\end{aligned} \tag{22}$$

$$\begin{aligned}
\parallel x_{n+(p-1)} - x_n \parallel &= \parallel (x_{n+(p-1)} - x_{n+(p-2)}) + \\
&+ (x_{n+(p-2)} - x_{n+(p-3)}) + \ldots + (x_{n+1} - x_n) \parallel \\
&\leq [(cd)^{(k+1)(p-2)+1} + (cd)^{(k+1)(p-3)+1} + \\
&\ldots + 1] \parallel x_{n+1} - x_n \parallel .
\end{aligned} \tag{23}$$

The inequality (21) because of (20), and (23) becomes

$$\parallel x_{n+p} - x_n \parallel \leq [\frac{1 - (cd)^{(k+1)p+1}}{1 - cd}](cd)^{(k+1)n} \parallel x_1 - x_0 \parallel . \tag{24}$$

Letting $n, p \to \infty$ in(24) and using (19) we obtain that the sequence $\{x_n\}$ is a Cauchy sequence in a Banach space X and as such it converges to some $x^* \epsilon B^*$ which, by the discussion made in the introduction, is a solution of equation (1).

This proves (i). The second part of the theorem is immediate from (22) and the inequality

$$\parallel x_n - x^* \parallel \leq \frac{(cd)^{(k+1)n}}{1 - cd} \parallel x_1 - x_0 \parallel, \quad n = 0, 1, 2, \ldots \tag{25}$$

which follows from (24) by letting $p \to \infty$. That completes the proof of the theorem.

Note that we can produce the "modified" version of (7) by introducing the iteration

$$\tilde{x}_{n+1} = T_{n,k}(\tilde{x}_0)F(\tilde{x}_n), \quad n = 0, 1, 2 \ldots \quad \text{for some} \quad \tilde{x}_0 \epsilon E$$

II. Examples.

Let $X = C$ the set of complex numbers equipped with the usual Euclidean norm $||$. Then $(X, ||)$ becomes a Banach space. Consider the equation

$$F(x) = x^3 - 5x^2 + 7x - 3 = 0. \tag{26}$$

Let $D = \bar{B}(.7, 1.2), x_0 = .7$. The linear operators $L_{n,2}(z)$ become

$$L_{n,2}(z)(x) = (3x_n^2 - 10x_n + 7)x + (3x_n - 5)(z - x_n)x$$

and the Newton-Kantorovich method for (26) gives

$x_0 = .7$
$x_1 = .840816$
$x_2 = .917578$
$x_3 = .957989$
$x_4 = .978781$
$x_5 = .989335$
$x_6 = .994653$
$x_7 = .997323$
$x_8 = .998661$
$x_9 = .99933$
$x_{10} = .999665$
$x_{11} = .999832$
$x_{12} = .999916$
$x_{13} = .999958$
$x_{14} = .999979$
$x_{15} = .999989$
$x_{16} = .999995$
$x_{17} = .999998$
$x_{18} = 1.$

The Newton-Kantorovich theorem guarantees the existence of a solution of (26) only after the 14^{th} iterate, since it can then easily be checked that

$$a = a(x_{14}) = .9981914 < 1, l = 4.000126.$$

It is well known, however, that Newton-Kantorovich can sometimes converge even if $x_0 \notin \bar{B}(x_{14}, r(x_{14}))$.

We can now observe that for $x_0 = .7, k = 2$ iteration (7) becomes a quadratic equation for every $n, n = 1, 2, 3 \ldots$.

For $n = 0$, (7) gives

$$-2.9x_1^2 + 5.53x_1 - 3.266 = 0$$

with solutions

$$s_1 = .9534482 \pm .465863i.$$

To apply the iteration (7) for $n = 1, 2 \ldots$ we choose

$$x_m = z_m = Rel(s_m), \quad m = 1, 2 \ldots.$$

That is, for $m = 0, z_1 = Rel(s_1) = .9534482$ and (7) becomes

$$-2.1396554z_2^2 + 4.2728098z_2 - 2.2282236 = 0$$

with solutions

$$s_2 = .9984808 \pm .2107835i.$$

The process will be terminated when $m = 4$ and the results can be tabulated as follows:

$z_0 = .7$
$z_1 = .9534482$
$z_2 = .9984808$
$z_3 = .99999982$
$z_4 = 1.$

We now observe that starting from the same initial guess, iteration (7) for $k = 2$ requires almost the one fourth of the number of iterations required from the same iteration (7) for $k = 1$ to obtain the solution $x^* = 1$ of equation (26).

Moreover, one can easily check that (16) is satisfied.

Finally, it is interesting to note that condition (19) is violated since,

$$c_n \rightarrow \infty \quad \text{as} \quad n \rightarrow \infty$$

and

$$d_n = 1, \quad n = 0, 1, 2, \ldots.$$

However, the sequence $\{z_n\}, n = 0, 1, 2, \ldots$ converges to the solution $x^* = 1$ of equation (26).

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