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WALD'S IDENTITY, BLACKWELL'S THEOREM, AND GUT AND JANSON'S THEOREM USING MARTINGALE TECHNIQUES

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Abstract The purpose of the present paper is to give different proofs for the well known Wald's identity using mainly martingale techniques. Two proofs for Blackwell's theorem [1] are given, the first one is due to Blackwell himself with some modifications using the strong law of large numbers, whereas the second proof is based on martingale techiques using a new result on the regularity (uniform integrability) of the stopped non-negative submartingales. This result is a generalization of a well known result of Neveu [2] for the regularity of the stopped martingales. Also, some interesting examples and remarks are presented. Moreover, the Gut and Janson's theorem [3] is reproduced and essentially simplified.

1. Introduction

Let $Y_1, Y_2, \ldots$ be a sequence of independent and identically distributed (i.i.d.) random variables defined on a probability space (Ω ,F,P), and let ν be a stopping time which is defined with respect to the sequence of σ-fields $F_n = \sigma(Y_1, Y_2, \ldots, Y_n), n \geq 1$.

Let $S_0 = 0$ and $S_n = Y_1 + Y_2 + \ldots + Y_n, n \geq 1$, be the "generalized" random walk. Then, the corresponding stopped random walk is $S_0 = 0$ and $S_{\nu\wedge n}, n = 1, 2, \ldots$, where $a \wedge b = \min(a, b)$ for any two real numbers a and b.

If ν is finite a.s., then the limit

$$\lim_{n\to+\infty} S_{\nu\wedge n} = S_\nu$$

exists and finite a.s., except the event $\{\nu = +\infty\}$, where S_ν is defined to be 0. Therefore, S_ν can be written in the following form:

$$S_\nu = \sum_{i=1}^{\infty} S_i \chi(\nu = i) = \sum_{i=1}^{\infty} Y_i \chi(+\infty > \nu \geq i),$$

where $\chi(A)$ stands for the indicator of the event A.

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Since Y_i and $\chi(\nu \geq i)$ are independent, and $P(\nu = +\infty) = 0$, it follows that Y_i and $\chi(+\infty > \nu \geq i)$ are also independent and $S_{\nu\wedge n}$ can be written as:

$$S_{\nu\wedge n} = \sum_{i=1}^{n} S_i\chi(\nu = i) = \sum_{i=1}^{n} Y_i\chi(+\infty > \nu \geq i).$$

Throughout the present paper we suppose that $P(\nu < +\infty) = 1$.

2. Wald's Identity

In this section we present two different proofs for the famous Wald's identity. The first proof is a modified and shortened form for the original proof given before, whereas the second one could be considered as a new proof and it mainly depends on martingale techniques.

Theorem 1.

If $E(\nu)$ and $E(Y_1) = a$ are finite, then the stopped random walk $S_{\nu\wedge n}$ converges in L^1 to its a.s. limit S_ν. As a consequence, we have

$$E(S_\nu) = aE(\nu).$$

This is known as the identity of Wald.

The First Proof:

We have

$$\begin{aligned} E(| S_\nu - S_{\nu\wedge n} |) &= E(| \sum_{i=n+1}^{\infty} Y_i\chi(+\infty > \nu \geq i) |) \leq \\ &\leq \sum_{i=n+1}^{\infty} E(| Y_i | \chi(+\infty > \nu \geq i)) = \\ &= \sum_{i=n+1}^{\infty} E(| Y_1 |)P(+\infty > \nu \geq i) = \\ &= E(| Y_1 |) \sum_{i=n+1}^{\infty} P(\nu \geq i). \end{aligned}$$

This tends to 0 as $n \to +\infty$ since $E(\nu) = \sum_{i=1}^{\infty} P(\nu \geq i) < +\infty$. Here, we have used the independence of Y_i and $\chi(+\infty > \nu \geq i)$. This is a consequence of the fact that $P(\nu = +\infty) = 0$ and that Y_i and $\chi(\nu \geq i)$ are independent.

Finally, since

$$E(S_{\nu\wedge n}) = \sum_{i=1}^{n} E(Y_i)P(\nu \geq i) = a\sum_{i=1}^{n} P(\nu \geq i),$$

we see by what we have proved above that

$$E(S_\nu) = \lim_{n\to+\infty} E(S_{\nu\wedge n}) = aE(\nu).$$

This proves the assertion.

For the second proof we need the following martingale theoretic result which was proved by J. Neveu [2].

Let $(X_n, F_n), n \geq 1$, be an integrable martingale with the differences $d_1 = X_1, d_i = X_i - X_{i-1}, i = 2, 3, \dots$, such that for each $i \geq 2$ we have $E(| d_i || F_{i-1}) \leq C$ a.s., where $C > 0$ is a constant. Consider the stopped martingale $(X_{\nu\wedge n}, F_n)$, where ν is a stopping time such that $E(\nu)$ is finite. The a.s. limit of the stopped martingale is:

$$X_\nu = \sum_{k=1}^{\infty} X_k\chi(\nu = k) = \sum_{k=1}^{\infty} d_k\chi(+\infty > \nu \geq k).$$

In both representations we define the a.s. limit X_ν to be equal to 0 on the event $\{\nu = +\infty\}$, which has 0 probability. We also have

$$X_{\nu\wedge n} = \sum_{k=1}^{n} d_k\chi(+\infty > \nu \geq k).$$

From the point of view of integration the terms $d_i\chi(+\infty \geq \nu > i)$ behave in the same manner as the random variables $d_i\chi(\nu \geq i)$, i=1,2,...

Lemma 1.

The stopped martingale $X_{\nu\wedge n}$ converges in L_1 to its a.s. limit X_ν. Especially, it follows that $E(X_\nu) = E(X_1)$.

Proof.

We have

$$E(| X_\nu - X_{\nu\wedge n} |) =$$

$$= E(| \sum_{i=n+1}^{\infty} d_i\chi(+\infty > \nu \geq i) |) \leq E(\sum_{i=n+1}^{\infty} | d_i | \chi(\nu \geq i)) =$$

$$= \sum_{i=n+1}^{\infty} E(E(| d_i || F_{i-1})\chi(\nu \geq i)) \leq C \sum_{i=n+1}^{\infty} P(\nu \geq i).$$

Here, we have used the assumption that $E(\mid d_i \mid\mid F_{i-1}) \leq C$ a.s. for $i \geq 2$ and that $\{\nu \geq i\} \epsilon F_{i-1}$. Now, if $n \to +\infty$ the righthand side tends to 0, since we supposed that $E(\nu) = \sum_{i=n}^{\infty} P(\nu \geq i) < +\infty$. Consequently, $X_{\nu \wedge n} \to X_\nu$ in L_1 as $n \to +\infty$. Especially, it follows that

$$E(X_\nu) = \lim_{n \to +\infty} E(X_{\nu \wedge n}) = E(X_1),$$

since $(X_{\nu \wedge n}, F_n)$ is a martingale.

This proves the assertion of the lemma.

The second proof of Theorem 1: We suppose that $a = E(Y_1)$ and $E(\nu)$ are finite. Consider the sequence $X_n = S_n - an$, which is a martingale with respect to the sequence $F_n = \sigma(Y_1, ..., Y_n), n \geq 1$, of σ-fields. The sequence $(S_{\nu \wedge n} - a(\nu \wedge n), F_n)$ is then the martingale $(S_n - an, F_n)$ stopped at ν. We have

$$S_{\nu \wedge n} - a(\nu \wedge n) = \sum_{i=1}^{n} (Y_i - a)\chi(+\infty > \nu \geq i)$$

where the differences of the original martingale are $Y_i - a, i \geq 1$. For (X_n, F_n) the assumptions of Lemma 1 are fulfilled. In fact, for $i \geq 2$, we have

$$E(\mid Y_i - a \mid\mid F_{i-1}) = E(\mid Y_i - a \mid) = E(\mid Y_1 - a \mid) = C,$$

which is finite and independent of i because the random variables $Y_i - a$ are identically distributed. Here, again we used the fact that $Y_i - a$ and $F_{i-1}, i \geq 2$, are independent. The limit of $S_{\nu \wedge n} - a(\nu \wedge n)$ is thus

$$S_\nu - a\nu = \sum_{i=1}^{\infty} (Y_i - a)\chi(+\infty > \nu \geq i)$$

in the sense of the L_1 convergence and a.s. convergence.

Consequently,

$$E(S_\nu - a\nu) = E(S_1 - a) = E(Y_1 - a) = 0.$$

Since $E(S_\nu)$ is finite, from the last equality, we obtain $E(S_\nu) = aE(\nu)$.
This proves the assertion.

Remarks.

(a) Wald's identity remains valid in the case when $a = E(Y_1) = \pm\infty$ and $E(\nu) < +\infty$. Namely, in this case we have

$$E(S_\nu) = \pm\infty = aE(\nu).$$

To prove this relation remark that the series

$$| Y_1 | + ... + | Y_\nu | = \sum_{i=1}^{\infty} | Y_i | \chi(+\infty > \nu \geq i)$$

absolutely converges. Thus,

$$S_\nu = \sum_{i=1}^{\infty} Y_i \chi(+\infty > \nu \geq i) = \sum_{i=1}^{\infty} (Y_i^+ - (Y_i^-) \chi(+\infty > \nu \geq i)$$

can be written in the form:

$$S_\nu = \sum_{i=1}^{\infty} Y_i^+ \chi(+\infty > \nu \geq i) - \sum_{i=1}^{\infty} Y_i^- \chi(+\infty > \nu \geq i).$$

Note that we have supposed that $E(\nu) < +\infty$ which implies $P(\nu < +\infty) = 1$. Consider the case when $a = E(Y_1) = +\infty$. The other case could be proved similarly. Then,

$$E(\sum_{i=1}^{\infty} Y_i^+ \chi(+\infty > \nu \geq i)) =$$

$$= \sum_{i=1}^{\infty} E(Y_i^+) P(\nu \geq i) = E(Y_1^+) E(\nu) = +\infty$$

and

$$E(\sum_{i=1}^{\infty} Y_i^- \chi(+\infty > \nu \geq i)) =$$

$$= \sum_{i=1}^{\infty} E(Y_i^-) P(\nu \geq i) = E(Y_1^-) E(\nu) < +\infty$$

since $Y_i = Y_i^+ - Y_i^-$ and now $E(Y_i^+) = +\infty$, whilst $E(Y_i^-) = E(Y_1^-) < +\infty$ and by supposition $E(\nu) < +\infty$.

If we suppose that $E(S_\nu)$ is finite then so must be

$$E(\sum_{i=1}^{\infty} Y_i^+ \chi(+\infty > \nu \geq i)),$$

since

$$E(\sum_{i=1}^{\infty} Y_i^+ \chi(+\infty > \nu \geq i)) = E(S_\nu) + E(\sum_{i=1}^{\infty} Y_i^- \chi(+\infty > \nu \geq i))$$

is finite, which is a contradiction. Suppose now that $E(S_\nu) = -\infty$. Then, from

$$\sum_{i=1}^{\infty} Y_i^+ \chi(+\infty > \nu \geq i) = S_\nu + \sum_{i=1}^{\infty} Y_i^- \chi(+\infty > \nu \geq i)$$

and from

$$E(\sum_{i=1}^{\infty} Y_i^- \chi(+\infty > \nu \geq i)) < +\infty$$

we deduce the validity of

$$E(\sum_{i=1}^{\infty} Y_i^+ \chi(+\infty > \nu \geq i)) = -\infty$$

which is again a contradiction. Finally, if we suppose that $E(S_\nu)$ does not exist, then so is

$$E(\sum_{i=1}^{\infty} Y_i^+ \chi(+\infty > \nu \geq i)),$$

which is a contradiction. These considerations prove that $E(S_\nu) = +\infty$.

(b) Wald's identity remains valid also in the case when $a = E(Y_1) \neq 0$ is finite and $E(\nu) = +\infty$. In [4] this case was considered and conditions were given to ensure that $E(S_\nu)$ is equal to $+\infty$. In section 4 of the present paper this case will also be considered and more restricted conditions are given to ensure that $E(S_\nu) = \pm\infty$ hold/Blackwell's Theorem/.

3. Stopping Times Related to a Sequence of i.i.d. Random Variables

To prove Blackwell's theorem we need to study the stopping times which are connected with a sequence of i.i.d. random variables.

The second assertion of this section will help us to prove Blackwell's theorem but at the same time it is interesting in its own right.

Lemma 2.

Let $Y_1, Y_2, ...$ be a sequence of i.i.d. random variables. Consider the σ-fields $F_n = \sigma(Y_1, ..., Y_n), n = 1, 2, ...$ and let ν be a stopping time with respect to the sequence $\{F_n\}_{n=1}^{\infty}$. Then, there is a Borel-measurable function $g : R^{\infty} \to \{+\infty, 1, 2, ...\}$ such that

$$\nu = g(Y_1, Y_2, ...)$$

Proof.

Introduce the notation

$$Y = (Y_1, Y_2, ...)$$

and for $n = 1, 2, ...$ let $Y^{(n)} = (Y_1, ..., Y_n)$.
Since $\{\omega : \nu = n\} \epsilon F_n = \sigma(Y_1, ..., Y_n)$, we deduce the existence of a Borel set $B^{(n)}$ of R^n such that

$$\{\nu = n\} = \{Y^{(n)} \epsilon B^{(n)}\}.$$

Denote by $B^{(n,\infty)}$ the Borel set $B^{(n)} \times R^{\infty}$. Then

$$\{\nu = n\} = \{Y \epsilon B^{(n,\infty)}\}.$$

The sets $B^{(n,\infty)}, n = 1, 2, \ldots$ are not necessarily pairwise disjoint. However, for $n \neq k$ the event

$$\{\omega : Y(\omega) \epsilon B^{(n,\infty)} \cap B^{(k,\infty)}\}$$

is empty since there is no ω for which at the same time $\nu(\omega) = n$ and $\nu(\omega) = k$. Thus if we define

$$C^{(n,\infty)} = B^{(n,\infty)} - \bigcup_{k=1, k\neq n}^{\infty} B^{(k,\infty)}, \quad n = 1, 2, ...$$

then these sets are pairwise disjoint and trivially we have

$$\{\nu = n\} = \{Y \epsilon C^{(n,\infty)}\}.$$

Finally, let

$$B^{(\infty)} = R^\infty - \bigcup_{n=1}^{\infty} C^{(n,\infty)}.$$

Then,

$$\{\nu = +\infty\} = \{Y \epsilon B^{(\infty)}\}.$$

This given, for $x = (x_1, x_2, ...) \epsilon R^\infty$, let

$$g(x) = \begin{cases} n & \text{if } x \epsilon C^{(n,\infty)} \quad n = 1, 2, ... \\ +\infty & \text{if } x \epsilon B^{(\infty)}. \end{cases}$$

Then, g is trivially a Borel measurable function on $(R^\infty, \mathcal{B}(R^\infty))$. Also, it trivially follows that $\nu = g(Y_1, Y_2, ...)$, which proves the assertion.

We also need the following very interesting

Theorem 2.

Let $Y_1, Y_2, ...$ be a sequence of i.i.d. random variables and let $F_n = \sigma(Y_1, ..., Y_n), n = 1, 2, ...$, be the corresponding sequence of σ-fields. Let ν be a stopping time with respect to the increasing sequence $\{F_n\}_{n=1}^{\infty}$ of σ-fields. Suppose that $P(\nu < +\infty) = 1$. Then

1 . every finite joint distribution of the sequence $Y_{\nu+1}, Y_{\nu+2}, ...$ is the same as that of sequence $Y_1, Y_2, ...$ Especially, the elements of the sequence $Y_{\nu+1}, Y_{\nu+2}, ...$ are independent and identically distributed and their distribution is the same as that of the Y_i's,

2 . the σ-fields, $\sigma(\nu)$ and σ $(Y_{\nu+1}, Y_{\nu+2}, ...)$ are independent,

3 . furthermore, the σ-fields $\sigma(Y_1, ..., Y_\nu)$ and $\sigma(Y_{\nu+1}, Y_{\nu+2}, ...)$ are independent.

Proof.

1. Consider $P((Y_{\nu+1}, Y_{\nu+2}, ..., Y_{\nu+k}) \epsilon B^{(k)}), B^{(k)} \epsilon \mathcal{B}(R^k)$, where $B^{(k)}$ is a k-dimensional Borel set. This is equal to

$$\sum_{j=1}^{\infty} P(\nu = j, (Y_{j+1}, ..., Y_{j+k}) \epsilon B^{(k)}) =$$

$$= \sum_{j=1}^{\infty} P(\nu = j) Q^k(B^{(k)}) = P(\nu < +\infty) Q^k(B^{(k)}) = Q^k(B^{(k)}),$$

where Q is the /common/distribution of the Y_i's and Q^k denotes the k-th Cartesian product of the measure Q. Note that $\{\nu = j\} \epsilon F_j$ and $\{(Y_{j+1}, ..., Y_{j+k}) \epsilon B^{(k)}\} \epsilon \sigma(Y_{j+1}, ..., Y_{j+k})$. Further F_j and $\sigma(Y_{j+1}, ..., Y_{j+k})$ are independent. This means that the members of the sequence $Y_{\nu+1}, Y_{\nu+2}, \ldots$ have the distribition Q and that they are independent.

2. Each element of the σ-field $\sigma(\nu)$ is of the form $\{\nu\epsilon G\}$ where G is an arbitrary subsequence of $\{+\infty, 1, 2, \ldots\}$. Consider $P(\nu\epsilon G, (Y_{\nu+1}, \ldots, Y_{\nu+k})\epsilon B^{(k)})$, where $B^{(k)}\epsilon\mathcal{B}(R^k)$ is an arbitrary Borel set. We can suppose that G does not contain the "index" $+\infty$, because by assumption $P(\nu < +\infty) = 1$. Now

$$P(\nu\epsilon G, (Y_{\nu+1}, \ldots, Y_{\nu+k})\epsilon B^{(k)}) =$$

$$= \sum_{j\epsilon G} P(\nu = j, (Y_{j+1}, \ldots, Y_{j+k})\epsilon B^{(k)}) =$$

$$= \sum_{j\epsilon G} P(\nu = j) P((Y_{j+1}, \ldots, Y_{j+k})\epsilon B^{(k)}),$$

since $\{\nu = j\}\epsilon F_j$ and $F_j = \sigma(Y_1, \ldots, Y_j)$ is independent of $\sigma(Y_{j+1}, \ldots, Y_{j+k})$. Further, as we proved in part 1 of the present proof,

$$P((Y_{j+1}, \ldots, Y_{j+k})\epsilon B^{(k)}) = Q^k(B^{(k)}),$$

where Q is the common distribution of the Y_i's and Q^k denotes the k-th direct product. Consequently,

$$P(\nu\epsilon G, (Y_{\nu+1}, \ldots, Y_{\nu+k})\epsilon B^{(k)}) =$$

$$= Q^k(B^{(k)}) \sum_{j\epsilon G} P(\nu = j) = P(\nu\epsilon G) Q^k(B^{(k)}).$$

Since $\sigma(Y_{\nu+1}, Y_{\nu+2}, \ldots) = \sigma(\bigcup_{k=1}^{\infty} \sigma(Y_{\nu+1}, \ldots, Y_{\nu+k}))$ and the preceding formula shows that the field $\bigcup_{k=1}^{\infty} \sigma(Y_{\nu+1}, \ldots, Y_{\nu+k})$ is independent of $\sigma(\nu)$, we see that $\sigma(\nu)$, and $\sigma(Y_{\nu+1}, Y_{\nu+2}, \ldots)$ are independent.

3. Consider $P((Y_1, \ldots, Y_\nu)\epsilon B^{(\nu)}, (Y_{\nu+1}, \ldots, Y_{\nu+k})\epsilon B^{(k)})$, where $B^{(\nu)}$ is a Borel set having dimension ν. We can consider this probability on the set $\{\nu < +\infty\}$ since $P(\nu = +\infty) = 0$. We have to show that this is equal to

$$P((Y_1, \ldots, Y_\nu)\epsilon B^{(\nu)}) P((Y_{\nu+1}, \ldots, Y_{\nu+k})\epsilon B^{(k)}).$$

Now

$$P((Y_1, \ldots, Y_\nu)\epsilon B^{(\nu)}, (Y_{\nu+1}, \ldots, Y_{\nu+k})\epsilon B^{(k)}) =$$

$$= \sum_{j=1}^{\infty} P(\nu = j, (Y_1, ..., Y_j)\epsilon B^{(j)}, (Y_{j+1}, ..., Y_{j+k})\epsilon B^{(k)}).$$

Note that $\{\nu = j, (Y_1, ..., Y_j)\epsilon B^{(j)}\}\epsilon F_j$, and that

$$\{(Y_{j+1}, ..., Y_{j+k}\epsilon B^{(k)}\}\epsilon\sigma(Y_{j+1}, ..., Y_{j+k})$$

and remark that the $\sigma-$fields F_j and $\sigma(Y_{j+1}, ..., Y_{j+k})$ are independent. Further, $P((Y_{j+1}, ..., Y_{j+k})\epsilon B^{(k)}) = Q^k(B^{(k)})$ independently of j. Therefore,

$$P((Y_1, ..., Y_\nu)\epsilon B^{(\nu)}, (Y_{\nu+1}, ..., Y_{\nu+k})\epsilon B^{(k)}) =$$

$$= Q^k(B^{(k)}) \sum_{j=1}^{\infty} P(\nu = j, (Y_1, ..., Y_j)\epsilon B^{(j)}) = Q^k(B^{(k)}) P((Y_1, ..., Y_{(\nu)})\epsilon B^{(\nu)}).$$

It follows that $\sigma(Y_1, ..., Y_\nu)$ and $\bigcup_{k=1}^{\infty} \sigma(Y_{\nu+1}, ..., Y_{\nu+k})$ are independent.
From this we deduce as above that $\sigma(Y_1, ..., Y_\nu)$ and $\sigma(Y_{\nu+1}, Y_{\nu+2}, ...)$ are independent.

This proves the assertion, cf. [5].

4. Sufficient and/or/Necessary Conditions for Convergence in L_1 (Blackwell's Theorem)

In this section we present two proofs for Blackwell's theorem. The first one is due to Blackwell himself (with some modifications) and uses the strong law of large numbers,whereas the second proof mainly depends on martingale setting and on a generalized result of the regularity of the stopped non-negative submartingales.

Theorem 3.

Let ν be an a.s. finite stopping time with respect to the σ-fields $F_n = \sigma(Y_1, Y_2, ..., Y_n), n \geq 1$. Suppose that $E(Y_1) = a \neq 0$ and $E(S_\nu)$ are finite. Then, $E(\nu)$ is finite, and consequently by Wald's assertion (Theorem 1) the stopped random walk $S_{\nu\wedge n}$ converges in L_1 to S_ν.

Proof.

Without loss of the generality we suppose that $E(Y_1) = a > 0$. Let $\nu_0 = 0, \nu_1 = \nu$ and consider the random variables $Y_{\nu+1}, Y_{\nu+2}, ...$ Now, by Theorem 2 they are independent and identically distributed/their common distribution is the same as that of the Y_i's/, and let us consider the stopping time

$$\nu_2 = g(Y_{\nu_1+1}, Y_{\nu_1+2}, ...),$$

where g is defined in Lemma 2. Further, consider the sequence $Y_{\nu_1+\nu_2+1}$, $Y_{\nu_1+\nu_2+2}, \dots$ Since $\nu_1+\nu_2$ is a stopping time and so by Theorem 2 the members of this sequence are independent and identically distributed, we see that the random variable

$$\nu_3 = g(Y_{\nu_1+\nu_2+1}, Y_{\nu_1+\nu_2+2}, \dots)$$

where g is defined by Lemma 2, is again a stopping time which is independent of ν_1 and ν_2. Repeat infinitely many times this procedure. Then, by Theorem 2, the sequence $\{\nu_k\}$ is an independent sequence of stopping times and by Lemma 2 the ν_k's are of the same distribution. Also, by Theorem 2 the differences

$$S_{\nu_0 \dots + \nu_{k+1}} - S_{\nu_0 + \dots + \nu_k}, \qquad k = 0, 1, 2, \dots$$

are independent and identically distributed. Their distribution is equal to that of $S_\nu = S_{\nu_1}$ and, consequently their expectation is $E(S_\nu)$. Now,

$$\frac{\nu_1 + \dots + \nu_k}{k} = \frac{S_{\nu_1} + \dots + \nu_k}{k} / \frac{S_{\nu_1} + \dots + \nu_k}{\nu_1 + \dots + \nu_k}.$$

Clearly, $\nu_1 + \dots + \nu_k \to +\infty$ as $k \to \infty$, since $\nu_i \geq 1, i \geq 1$. By the strong law of large numbers it follows that

$$\frac{S_{\nu_1} + \dots + \nu_k}{\nu_1 + \dots + \nu_k} \to E(Y_1) = a > 0 \quad a.s. \ as \ k \to +\infty$$

and that

$$\frac{S_{\nu_1} + \dots + \nu_k}{k} \to E(S_\nu) \quad a.s. \ as \ k \to +\infty.$$

Consequently,

$$\frac{\nu_1 + \dots + \nu_k}{k} \to a^{-1} E(S_\nu) \quad a.s. \ as \ k \to +\infty.$$

By the converse of the Kolmogorov strong law of large numbers, it follows that $E(\nu)$ is finite and

$$E(\nu) = a^{-1} E(S_\nu).$$

This proves our assertion/cf.[5]/.

Our method of proof of Blackwell's theorem permits us to prove the following extension of Wald's identity:

Remark.

If $a = E(Y_1) > 0$ is finite and $E(\nu) = +\infty$ then necessarily $E(S_\nu) = +\infty$. To prove this, consider the same random variables as in the proof of the preceding theorem. Then, we have

$$\frac{\nu_1 + \ldots + \nu_k}{k} \frac{S_{\nu_1} + \ldots + \nu_k}{\nu_1 + \ldots + \nu_k} = \frac{S_{\nu_1} + \ldots + \nu_k}{k}.$$

Since, by the strong law of large numbers, the first factor on the left hand side tends to $+\infty$ and the second to $a > 0$, we see that the a.s. limit of the right hand side is equal to $+\infty$. This means that $E(S_\nu) = +\infty$.

Blackwell's theorem says that if $(S_{\nu\wedge n}, F_n)$ is an integrable submartingale/supermartingale/and $E(Y_1) = a > 0$, then from the fact that $E(S_\nu)$ is finite it follows that $E(\nu) < +\infty$. In other words, on the basis of Theorem 1, $(S_{\nu\wedge n}, F_n)$ converges in L_1 to S_ν.

Now, we give a new proof for this fact by helps of a submartingale convergence theorem. The following theorem would be considered as a generalization of a well-known theorem of J.Neveu cf.[2].

Theorem 4.

Let (X_n, F_n) be a non-negative submartingale and let ν be a stopping time. The stopped submartingale converges in L_1 and $\lim_{n\to+\infty} X_n = 0$ a.s. on $\{\nu = +\infty\}$ if and only if the conditions

$$\int_{\{\nu<+\infty\}} X_\nu dP < +\infty, \lim_{n\to+\infty} \int_{\{\nu>n\}} X_n dP = 0.$$

are satisfied.

Proof.

Necessity. If the sequence $(X_{\nu\wedge n}, F_n)$ converges in L_1 to a limit Y, which is equal to 0 on $\{\nu = +\infty\}$, then $Y = X_\nu$ on $\{\nu < +\infty\}$ because on this event the a.s. limit of $X_{\nu\wedge n}$ is X_ν. Moreover, we have

$$\int_{\{\nu<+\infty\}} X_\nu dP = \int_\Omega Y dP < +\infty,$$

and, as $n \to +\infty$,

$$\int_{\{\nu>n\}} X_n dP = \int_{\{\nu>n\}} X_{\nu\wedge n} dP \to \int_{\{\nu=+\infty\}} Y dP = O,$$

using the L_1-convergence of $X_{\nu\wedge n}$ to Y.

Sufficiency. To prove that the conditions are sufficient we remark that the non-negative submartingale $\{X_{\nu\wedge n}\}$ is bounded L_1 in since when $n \to +\infty$ we have

$$E(X_{\nu\wedge n}) = \int_{\{\nu\leq n\}} X_\nu dP + \int_{\{\nu>n\}} X_n dP \to \int_{\{\nu<+\infty\}} X_\nu dP < +\infty.$$

Thus $\{X_{\nu\wedge n}, F_n\}$ converges a.s. and the limit $\lim_{n\to+\infty} X_n$ exists on $\{\nu = +\infty\}$, too. Especially, this limit equals 0 since by the Fatou's lemma

$$\int_{\{\nu=+\infty\}} \lim_{n\to+\infty} X_n dP \leq \lim \inf_{n\to+\infty} \int_{\{\nu=+\infty\}} X_n dP \leq$$

$$\leq \lim_{n\to+\infty} \int_{\{\nu>n\}} X_n dP = 0.$$

Therefore, the random variable X_ν can be defined everywhere by putting $X_\nu = \lim_{n\to+\infty} X_n = 0$ on $\{\nu = +\infty\}$. The convergence of $(X_{\nu\wedge n}, F_n)$ to X_ν in L_1 can be proved in the following way:

$$E(| X_{\nu\wedge n} - X_\nu |) = \int_{\{\nu>n\}} | X_{\nu\wedge n} - X_\nu | dP \leq$$

$$\leq \int_{\{\nu>n\}} X_{\nu\wedge n} dP + \int_{\{\nu>n\}} X_\nu dP =$$

$$= \int_{\{\nu>n\}} X_n dP + \int_{\{\nu>n\}} X_\nu dP \to 0 + \int_{\{\nu=+\infty\}} X_\nu dP = 0.$$

This proves the assertion.

Now, we are in the position to prove Blackwell's theorem in another manner using martingale techniques.

Theorem 5.

Let ν be a stopping time such that $P(\nu < +\infty) = 1$. Also, we suppose that $E(Y_1) = a > 0$ and $E(S_\nu)$ are finite. Then the stopped submartingale $(S_{\nu\wedge n}, F_n)$, converges in L_1 to S_ν if and only if $E(\nu) < +\infty$.

Proof.

Suppose that $S_{\nu\wedge n}$ converges in L_1. Then, its L_1 limit is necessarily equal to S_ν, since S_ν is the a.s. limit of $S_{\nu\wedge n}$ when $n \to +\infty$. Consequently, $| S_\nu |$ is integrable. We have

$$| S_{\nu\wedge n} |= \sum_{i=1}^{n} | S_i | \chi(\nu = i)+ | S_n | \chi(\nu > n)$$

and this converges to $| S_\nu |$ in L_1. At the same time, the sequence $\sum_{i=1}^{n} | S_i | \chi(\nu = i)$ converges increasingly to $| S_\nu |$ so that by the monotone convergence theorem, we also conclude that

$$E(| S_\nu - \sum_{i=1}^{n} S_i\chi(\nu = i) |) \downarrow 0 \quad (n \to +\infty).$$

From these we conclude that $E(| S_n | \chi(\nu > n)) \to 0$ as $n \to +\infty$. This means that we also have $E(S_n^+\chi(\nu > n)) \to 0$ as $n \to +\infty$.

Consider the non-negative submartingale $(S_{\nu\wedge n}^+, F_n)$. By the preceding theorem $E(S_\nu^+\chi(\nu < +\infty)) < +\infty$ and $E(S_n^+\chi(\nu > n)) \to 0$ together imply that $(S_{\nu\wedge n}^+, F_n)$ converges a.s. and in L_1 to S_ν^+. It is easy to see that

$$0 = \lim_{n\to+\infty} E(S_n^+\chi(\nu > n)) = \lim_{n\to+\infty} E(S_{\nu\wedge n}^+) - E(S_\nu^+),$$

since

$$E(S_n^+\chi(\nu > n)) = E(S_{\nu\wedge n}^+) - \int_{\{\nu\leq n\}} S_\nu^+ dP.$$

The expectation $E(S_\nu^+)$ being finite we deduce that the increasing limit $\lim_{n\to+\infty} E(S_{\nu\wedge n}^+)$ is finite and equals $E(S_\nu^+)$. Since $E(S_{\nu\wedge n}) = aE(\nu \wedge n) > 0$, we conclude that

$$\lim_{n\to+\infty} E(S_{\nu\wedge n}^+) \geq \sup_{n\geq 1} E(| S_{\nu\wedge n} |)/2,$$

which implies that $\sup_{n>1} E(| S_{\nu\wedge n} |) < +\infty$.

Now, to prove that $E(\nu) < +\infty$ remark that

$$| E(S_{\nu\wedge n}) |\leq E(| S_{\nu\wedge n} |) \leq \sup_{k\geq 1} E(| S_{\nu\wedge k} |) < +\infty.$$

By means of Wald's identity, we have $E(S_{\nu\wedge n}) = aE(\nu \wedge n)$ and so

$$| E(S_{\nu\wedge n}) |= aE(\nu \wedge n) \leq \sup_{k>1} E(| S_{\nu\wedge k} |) < +\infty.$$

Now, let $n \to +\infty$, then

$$aE(\nu \wedge n) \to aE(\nu)$$

and so we have

$$aE(\nu) \leq \sup_{k\geq 1} E(| S_{\nu\wedge k} |) < +\infty$$

and since $a > 0$ and finite, it follows that $E(\nu) < +\infty$.

Conversely, if $E(\nu) < +\infty$, then $a = E(Y_1) > 0$ being finite, by Theorem 1 we conclude that $S_{\nu\wedge n}$ converges in L_1 to S_ν. This proves our assertion.

5. Gut and Janson's Theorem

Here, we also reproduce the original theorem of Gut and Janson [3] and, as we shall see, the proof is essentially simplified.

Let $\nu_0 = 0, \nu_1 = \nu$ and

$$\nu_2 = g(Y_{\nu_1+1}, Y_{\nu_1+2}, ...), ...,$$

$$\nu_k = g(Y_{\nu_{k-1}+1}, Y_{\nu_{k-1}+2}, ...)$$

be the random variables defined by the function g of Lemma 2. These by Theorem 2 are independent, identically distributed and positive integer-valued having finite expectation $E(\nu)$.

Let τ_n be the stopping time defined by the formula:

$$\tau_n = \min(k : \nu_1 + ... + \nu_k \geq n).$$

Then, $1 \leq \tau_n \leq n$, since $\nu_1, \nu_2, ...$ are not less than 1.

We have the following renewal theoretic result.

Lemma 3.

If $n \to +\infty$ then

$$\tau_n/n \to 1/E(\nu) \qquad a.s.$$

Proof.

Remark that $\tau_n \to +\infty$ a.s. when $n \to +\infty$. Also,

$$\nu_1 + ... + \nu_{\tau_n - 1} < n \leq \nu_1 + ... + \nu_{\tau_n}$$

Dividing by τ_n and using the strong law of large numbers for a random number of random variables we see that:

$$\lim_{n \to +\infty} \frac{n}{\tau_n} = E(\nu) \qquad a.s.$$

This proves the assertion of the lemma.

The following assertion gives an estimate of the distribution of the random variable $\nu_1 + ... + \nu_{\tau_n} - n$.

Lemma 4.

For $j = 0, 1, 2, ...,$ we have

$$P(\nu_1 + ... + \nu_{\tau_n} = n + j) \leq P(\nu_1 \geq j + 1).$$

Proof.

For every fixed $j = 0, 1, 2, ...$ we have

$$P(\nu_1 + ... + \nu_{\tau_n} = n + j) =$$

$$= \sum_{k=1}^{n} P(\nu_1 + ... + \nu_k = n + j, \tau_n = k) =$$

$$= \sum_{k=1}^{n} \sum_{\ell=1}^{n-1} P(\nu_1 + ... + \nu_{k-1} = \ell, \nu_k = n + j - \ell, \tau_n = k) \leq$$

$$\leq \sum_{k=1}^{n} \sum_{\ell=1}^{n-1} P(\nu_1 + ... + \nu_{k-1} = \ell) P(\nu_k = n + j - \ell) =$$

$$= \sum_{k=1}^{n} \sum_{\ell=1}^{n-1} P(\nu_1 = n + j - \ell) P(\nu_1 + ... + \nu_{k-1} = \ell),$$

since ν_k and $\nu_1 + ... + \nu_{k-1}$ are independent and ν_k has the same distribution as ν_1. Since for fixed ℓ the events $\{\nu_1 + ... + \nu_{k-1} = \ell\}, k = 2, 3, ...$ are disjoint, we have the folloving estimate

$$P(\nu_1 + ... + \nu_{\tau_n} = n + j) \leq$$

$$\leq \sum_{\ell=1}^{n-1} P(\nu_1 = n + j - \ell) \sum_{k=1}^{n} P(\nu_1 + ... + \nu_{k-1} = \ell) \leq$$

$$\leq \sum_{\ell=1}^{n-1} P(\nu_1 = n + j - \ell) \leq P(\nu_1 \geq j + 1).$$

This proves the assertion.

We are now in the position to prove the following:

Theorem 6.

Let $\{S_{\nu \wedge n}, F_n\}$ be the stopped random walk and suppose that $E(S_\nu)$ and $E(\nu)$ are finite. Then, necessarily $E(Y_1)$ is finite.

Proof.

We can suppose that $E(S_\nu) = 0$, otherwise we substitute Y_i by $Y_i - E(S_\nu)/E(\nu)$. Let $\nu_1, \nu_2, ...$ and τ_n be defined as in the preceding two lemmas. As

$$S_{\nu_1} + ... + \nu_k = (S_{\nu_1} - S_0) + (S_{\nu_1+\nu_2} - S_{\nu_1}) + ... + (S_{\nu_1+...+\nu_k} - S_{\nu_1+..+\nu_{k-1}})$$

and the members on the right hand side are independent and identically distributed with expectation $E(S_\nu) = 0$, we see by the strong law of large numbers that

$$\frac{S_{\nu_1+...+\nu_k}}{k} \to 0 \qquad a.s.$$

as $k \to +\infty$. Also, by the strong law of large numbers

$$\frac{S_{\nu_1} + ... + \nu_{\tau_n}}{\tau_n} \to 0 \qquad a.s.$$

as $n \to +\infty$, since $\tau_n \to +\infty$ as $n \to +\infty$. This and the preceding limit relation together imply that

$$\frac{S_{\nu_1} + ... + \nu_{\tau_n}}{n} \to 0 \qquad a.s.$$

as $n \to +\infty$ since by Lemma 3 we have $\tau_n/n \to (E(\nu))^{-1}$.

Now, we show that $S_n/n \to 0$ in probability when $n \to +\infty$. In fact, we have

$$P(|S_n| \geq 2n\epsilon) \leq$$
$$\leq P(\{|S_{\nu_1+\ldots+\nu_{\tau_n}}| \geq n\epsilon\} U \{|S_{\nu_1+\ldots+\nu_{\tau_n}} - S_n| \geq n\epsilon\}) \leq$$
$$\leq P(|S_{\nu_1} + \ldots + \nu_{\tau_n}| \geq n\epsilon) + P(|S_{n+k} - S_n| \geq n\epsilon,$$
$$\text{for some } k \leq \nu_1 + \ldots + \nu_{\tau_n} - n) \leq$$
$$\leq P(|S_{\nu_1} + \ldots + \nu_{\tau_n}| \geq n\epsilon) + P(|S_{n+k} - S_n| \geq n\epsilon,$$
$$\text{for some } k \leq \nu_1 + \ldots + \nu_{\tau_n} - n, \nu_1 + \ldots + \nu_{\tau_n} - n \leq j)$$
$$+ P(\nu_1 + \ldots + \nu_{\tau_n} - n > j) \leq$$
$$\leq P(|S_{\nu_1} + \ldots + \nu_{\tau_n}| \geq n\epsilon) + P(\max_{1 \geq \ell \geq j} |S_\ell| \geq n\epsilon) + \sum_{k=j+1}^{\infty} P(\nu_1 \geq k),$$

by lemma 4. From these we see that:

$$\lim \sup_{n \to +\infty} P(|\frac{S_n}{n}| \geq 2\epsilon) \leq \sum_{k=j+1}^{\infty} P(\nu_1 \geq k)$$

as $n \to +\infty$. Since j was arbitrarily chosen but fixed and $E(\nu) < +\infty$ we get:

$$\lim \sup_{n \to +\infty} P(|\frac{S_n}{n}| \geq 2\epsilon) = 0.$$

In the next step of the proof we show that

$$\frac{1}{n} \max_{1 \leq k \leq n} |S_k| \to 0$$

in probability when $n \to +\infty$. If the random variables $Y_1, Y_2, \ldots$ are independent identically and symmetrically distributed then this fact follows from Paul Levy's inequality according to which $P(\max_{1 \leq k \leq n} |S_k| \geq n\epsilon) \leq 2P(|S_n| \geq n\epsilon)$ and we deduce that

$$P(\frac{1}{n} \max_{1 \leq k \leq n} |S_k| \geq \epsilon) \to 0$$

as $n \to +\infty$. In the general case let $Y_1', Y_2', \ldots$ be i.i.d. random variables and let us suppose that this sequence is independent of the sequence $Y_1, Y_2, \ldots$ and has the same distribution. Then the sequence $Y_1 - Y_1', Y_2 - Y_2', \ldots$ is a sequence of independent , identically distributed and symmetrical random variables. Denote $S_k' = Y_1' + \ldots + Y_k', k = 1, 2, \ldots$ Then from the preceding limit relation

$$P(\frac{1}{n} \max_{1 \leq k \leq n} |S_k - S_k'| \geq \epsilon) \leq$$

$$\leq 2P(|S_n - S'_n| \geq n\epsilon) \leq 4P(|S_n| \geq n\frac{\epsilon}{2}) \to 0$$

as $n \to +\infty$. But by the symmetrization inequality

$$\frac{1}{2}P(\max_{1\leq k\leq n} |S_k - m(S_k)| \geq n\epsilon) \leq P(\frac{1}{n}\max_{1\leq k\leq n} |S_k - S'_k| \geq \epsilon) \to 0,$$

where $m(S_k)$ denotes the median of S_k. Note that

$$\frac{\max_{1\leq k\leq n} |m(S_k)|}{n} \to 0$$

as $n \to +\infty$. To show this let us remark that $m(cX) = cm(X)$ for arbitrary $c > 0$ and that $P(a < X < b) > \frac{1}{2}$ implies $a \leq m(X) \leq b$. By what we already proved we have

$$P(|\frac{S_n}{n}| < \epsilon) > \frac{1}{2},$$

if $n \geq n_0(\epsilon)$. Thus,

$$|\frac{m(S_k)}{n}| = |m(\frac{S_k}{n})| = |m(\frac{S_k}{k})| \frac{k}{n} \leq |m(\frac{S_k}{k})| \leq \epsilon,$$

whenever $k \geq n_0(\epsilon)$. For $k \leq n_0 = n_0(\epsilon)$ we have $S_k/n \to 0$ a.s. and consequently for sufficiently large n we have

$$P(|\frac{S_k}{n}| < \epsilon) > \frac{1}{2},$$

which shows that

$$\frac{|m(S_k)|}{n} \leq \epsilon.$$

Therefore,

$$\max_{1\leq k\leq n} \frac{|m(S_k)|}{n} \to 0$$

as $n \to +\infty$. Let $n_1 = n_1(\epsilon)$ be so large that the inequality

$$\max_{1\leq k\leq n} \frac{|m(S_k)|}{n} \leq \epsilon/2$$

be satisfied. Then, by the preceding limit relation

$$P(\frac{1}{n}\max_{1\leq k\leq n} | S_k |\geq \epsilon/2) \to 0$$

as $n \to +\infty$ whatever be $\epsilon > 0$. This implies that for $n \geq n_0$, where n_0 is appropriately chosen, we have

$$P(\max_{1\leq k\leq n} | S_k |> n) < \frac{1}{2}.$$

For $k \geq n_0$, we have

$$P(\min_{1\leq j\leq k} | S_j |> k) \geq P(\{| Y_1 |> 2k\} \cap \{\max_{1<j\leq k} | S_j - S_1 |\leq k\}) =$$

$$= P(| Y_1 |> 2k)P(\max_{1\leq j\leq k-1} | S_j |\leq k) \geq \frac{1}{2}P(| Y_1 |> 2k).$$

On the other hand, for all k, we have

$$P(\min_{1\leq j\leq k} | S_j |> k) = P(\{\min_{1\leq j\leq k} | S_j |> k\} \cap \{\nu > k\})+$$

$$+P(\{\min_{1\leq j\leq k} | S_j |> k\} \cap \{\nu \leq k\}) \leq$$

$$\leq P(\nu > k) + P(| S_\nu |> k).$$

Summation finally yields

$$\frac{1}{2}\sum_{k=n_0}^{\infty} P(| Y_1 |> 2k) \leq$$

$$\leq \sum_{k=1}^{\infty} P(\nu > k) + \sum_{k=1}^{\infty} P(| S_\nu |> k) \leq E(\nu) + E(| S_\nu |) < +\infty,$$

and thus $E(| Y_1 |) < +\infty$. This proves the assertion.

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