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# ON TWO-SIDED INEQUALITIES FOR STOPPED RANDOM WALKS IN ORLICZ-SPACES

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Abstract. The aim of the present note is to generalize the work done by N.L. Bassily, S. Ishak, J. Mogyoródi in [1] and by J. Mogyoródi in [2]. Namely, a modified form of the well known Burkholder-Davis-Gundy inequality [3] and [4] as well as a Rosenthal type inequality c.f. [2] are used to study the behaviour of the almost sure limit $S_\nu - a\nu$ of the generalized stopped random walk in Orlicz-spaces. Here $S_n = Y_1 + Y_2 + \ldots + Y_n, n \geq 1$, where $Y_i, i \geq 1$, is a sequence of i.i.d. random variables with $E(Y_i) = a$ finite, $i \geq 1$. Also $\nu$ is a stopping time with respect to the increasing sequence of $\sigma$-fields $F_n = \sigma(Y_1, Y_2, \ldots, Y_n), n \geq 1$ such that $P(\nu < +\infty) = 1$. Estimates for the supremum of the stopped partial sums of i.i.d. random variables with zero mean in Orlicz-spaces are given.

## 1. Introduction

For $x \geq 0$ let us consider the function $\Phi(x) = \int_0^x \phi(t)dt$, where the integrand $\phi(t)$ is right-continuous and increasing with $\phi(0) = 0$ and $\lim_{t\to\infty} \phi(t) = +\infty$. Then, as it is known, $\Phi(x)$ is continuous, increasing and convex function of $x$, having the property that $\Phi(x)/x$ increases and

$$\lim_{x\downarrow 0} \frac{\Phi(x)}{x} = 0, \quad \lim_{x\uparrow+\infty} \frac{\Phi(x)}{x} = +\infty.$$

The function $\Phi$ is called a Young function. Consider the generalized inverse of $\phi$, i.e. let $\psi(t) = \phi^{-1}(t)$. $\psi(t)$ has the same properties as $\phi(t)$, i.e. it is non-decreasing and right-continuous such that $\psi(0) = 0$ and $\lim_{t\to+\infty} \psi(t) = +\infty$.

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Consequently, the function $\Psi(x) = \int_0^x \psi(t)dt$ is also Young function. $\Psi(x)$ is called the Young function conjugate to $\Phi(x)$. $\Phi$ and $\Psi$ mutually determine each other.

Examples for pairs of conjugate Young functions are the following:

a) for $p > 1$

$$\Phi(x) = \frac{x^p}{p}, \quad \Psi(x) = \frac{x^q}{q},$$

where $q$ is the conjugate power of $p$, i.e. $p^{-1} + q^{-1} = 1$;

b) $\Phi(x) = e^x - x - 1, \Psi(x) = (x+1)\log(x+1) - x$.

For every Young function $\Phi$ we define its power $p$ by the formula

$$p = \sup_{x>0} \frac{x\phi(x)}{\Phi(x)}.$$

In case a) both Young functions have finite power, namely, $p > 1$ and $q > 1$, respectively. In case b) the Young function $\Phi(x)$ has infinite power whilst its conjugate $\Psi(x)$ has a finite one.

We say that the random variable $X$ defined on the probability space $(\Omega, A, P)$ belongs to Orlicz space $L^\Phi = L^\Phi(\Omega, A, P)$, if there exists a number $a > 0$ such that $E(\Phi(a^{-1} \mid X \mid)) \leq 1$. In this case we define

$$\| X \|_\Phi = \inf(a > 0 : E(\Phi(a^{-1} \mid X \mid)) \leq 1).$$

It can be easily proved that $\| \cdot \|_\Phi$ is a seminorm. More about the Young functions and Orlicz spaces can be read e.g. in [5].

Given a martingale $(X_n, F_n), n \geq 0$ with $X_0 = 0$, let $d_0 = 0$ and $d_i = X_i - X_{i-1}, i \geq 1$, be its difference sequence. The quadratic variation of the martingale is defined by

$$V = V(X) = (\sum_{i=0}^{\infty} d_i^2)^{1/2}.$$

Let

$$X^* = \sup_{n\geq 1} \mid X_n \mid,$$

be the maximal function of the martingale $(X_n, F_n), n \geq 0$.

Let $\Phi(x)$ be a Young function with finite power $p$. Given a martingale $(X_n, F_n), n \geq 0$, the Burkholder-Davis-Gundy inequality says that

$$c_\Phi E(\Phi(V)) \leq E(\Phi(X^*)) \leq C_\Phi E(\Phi(V)),$$

where $c_\Phi > 0$ and $C_\Phi > 0$ are constants depending only on $\Phi$.
This inequality is meant in the generalized sense, i.e. the left and the right hand sides are finite if and only if so is $E(\Phi(X^*))$. In this case the sequence $\{X_n\}_{n=0}^{\infty}$ is uniformly integrable and thus there exists a random variable $X$ such that $X_n = E(X \mid F_n)$ a.s. for every $n \geq 1$, e.g. we can take for $X$ the a.s. limit of $\{X_n\}_{n=0}^{\infty}$, which exists under the condition $E(\Phi(X^*)) < +\infty$. Moreover, $X = limn \to +\infty$ a.s. $X_n = X_\infty \epsilon L^\Phi$. Suppose that $\| V \|_\Phi < +\infty$. In this case we say that the a.s. limit $X$ of $\{X_n\}_{n=0}^{\infty}$ belongs to the Hardy space $H_\Phi$ generated by the Young function $\Phi$ and we define

$$\| X \|_{H_\Phi} = \| V \|_\Phi$$

Thus, $\| \cdot \|_{H_\Phi}$ is a seminorm.

Suppose that $\Phi$ has finite power $p$ and that $\| V \|_\Phi$ as well as $\| X^* \|_\Phi$ are positive and finite. Then the above Burkholder-Davis-Gundy inequality implies that

$$\| X^* \|_\Phi \leq \rho_2 \| V \|_\Phi \tag{1}$$

holds with some constant $\rho_2 > 0$ depending only on $\Phi$. In fact, $E(\Phi(V/ \| V \|_\Phi)) = 1$, and if $a \uparrow +\infty$ then $\Phi(V/a \| V \|_\Phi)$ tends decreasingly to 0. Consequently, there exists a $\rho_2 > 0$ such that $C_\Phi E(\Phi(V/\rho_2 \| V \|_\Phi)) = 1$ is satisfied. It follows that

$$E(\Phi(X^*/\rho_2 \| V \|_\Phi)) \leq C_\Phi E(\Phi(V/\rho_2 \| V \|_\Phi)) = 1$$

which implies (1). Similarly we can easily prove that

$$\| V \|_\Phi \leq \rho_1 \| X^* \|_\Phi \tag{2}$$

holds. Namely, again by the Burkholder-Davis-Gundy inequality we have

$$c_\Phi E(\Phi(V)) \leq E(\Phi(X^*))$$

and

$$E(\Phi(X^*/ \| X^* \|_\Phi)) = 1.$$

So, if $a \uparrow +\infty$ then $\Phi(X^*/a \parallel X^* \parallel_\Phi)$ tends decreasingly to 0. Consequently, there exists a $\rho_1 > 0$ such that $\frac{1}{c_\Phi}E(\Phi(X^*/\rho_1 \parallel X^* \parallel_\Phi)) = 1$ is satisfied. It follows that

$$E(\Phi(V/\rho_1 \parallel X^* \parallel_\Phi)) \leq \frac{1}{c_\Phi}E(\Phi(X^*/\rho_1 \parallel X^* \parallel_\Phi)) = 1,$$

which implies (2). Comparing (1) and (2) we get

$$\frac{1}{\rho_1} \parallel V \parallel_\Phi \leq \parallel X^* \parallel_\Phi \leq \rho_2 \parallel V \parallel_\Phi, \tag{3}$$

which is the Burkholder-Davis-Gundy inequality in its seminorm form (4).

Inequality (3) says that the almost sure limit $X$ of the martingale $(X_n, F_n)_{n=0}^\infty$ belongs to the Hardy space $H_\Phi$ if and only if the maximal function $X^*$ belongs to $L^\Phi$.

## 2. Inequalities for the Stopped Random Walks.

Let $Y_1, Y_2, \ldots$ be independent and identically distributed random variables (i.i.d.) and let $S_0 = 0$ and $S_n = Y_1 + \ldots + Y_n, n \geq 1$, be the corresponding random walk. We shall suppose that $E(Y_1) = a$ is finite.

Consider the $\sigma$-field $F_n = \sigma(Y_1, \ldots, +Y_n), n \geq 1$, and let $F_0 = (\phi, \Omega)$ be the trivial $\sigma$-field. Then the sequence $\{S_n - na, F_n\}, n \geq 0$, is a martingale with respect to the increasing sequence $\{F_n\}_{n=0}^\infty$.

Given two real numbers $a$ and $b$ we introduce the notation $a \wedge b = \min(a, b)$. If $\nu$ is a stopping time with respect to $\{F_n\}_{n=0}^\infty$ such that $P(\nu < +\infty) = 1$, then the sequence $(S_{\nu\wedge n} - a(\nu \wedge n), F_n)$ is the martingale $\{S_n - na, F_n\}_{n=0}^\infty$ stopped at the moment $\nu$. The almost sure limit of $\{S_{\nu\wedge n} - a(\nu \wedge n)\}_{n=0}^\infty$ exists on the event $\{\nu < +\infty\}$ whilst we define it to be equal to 0 on the null event $\{\nu = +\infty\}$. So, the limit $\lim_{n\to+\infty}(S_{\nu\wedge n} - a(\nu \wedge n))$ is equal to $S_\nu - a\nu$ on the event $\{\nu < +\infty\}$ and equals 0 on $\{\nu = +\infty\}$. This limit can be expressed in the following two equivalent forms:

$$\sum_{i=1}^{\infty}(Y_i - a)\chi(+\infty > \nu \geq i) = \sum_{n=1}^{\infty}(S_n - na)\chi(\nu = n),$$

where $\chi(A)$ stands for the indicator of the event $A$.

The differences of the martingale $(S_{\nu\wedge n} - a(\nu\wedge n), F_n)_{n=0}^\infty$ are the following: $d_0 = 0, d_i = (Y_i - a)\chi(\nu \geq i),\ i \geq 1$. Note that $Y_i - a$ and $\chi(\nu \geq i)$ are independent.

In theorem 1, the necessary and sufficient condition for the a.s. limit $S_\nu - a\nu$ of the stopped random walk to belong to $H_\Phi$ is given.

**Theorem 1.** The a.s. limit $S_\nu - a\nu$ of the martingale $\{S_{\nu\wedge n} - a(\nu \wedge n), F_n\}_{n=0}^{\infty}$ belongs to $H_\Phi$, where $\Phi$ has finite power $p$ if and only if

$$(*) \qquad \| V \|_\Phi = \| (\sum_{i=1}^{\infty} (Y_i - a)^2 \chi(\nu \geq i))^{1/2} \|_\Phi < +\infty.$$

Moreover, if $\Psi$, the conjugete Young function has a finite power $q$, then

$$q^{-1}\frac{1}{\rho_1} \| V \|_\Phi \leq \| S_\nu - a\nu \|_\Phi \leq$$
$$\leq \| \sup_{n\geq 1} | S_{\nu\wedge n} - a(\nu \wedge n) | \|_\Phi \leq \rho_2 \| V \|_\Phi,$$

where $\rho_1$ and $\rho_2$ are the constants defined in (3).

**Proof.** The first part is an immediate consequence of the martingale-theoretic results cited in the preceding section.

Further, if the power $q$ of $\Psi$ is also finite then by the Doob inequality (see [6],[7]) we have

$$\| \sup_{n\geq 1} | S_{\nu\wedge n} - a(\nu \wedge n) | \|_\Phi \leq q \sup_{n\geq 1} \| S_{\nu\wedge n} - a(\nu \wedge n) \|_\Phi =$$
$$= q \| S_\nu - a\nu \|_\Phi .$$

From this it follows that

$$\frac{1}{\rho_1} q^{-1} \| V \|_\Phi \leq \| S_\nu - a\nu \|_\Phi \leq$$
$$\leq \| \sup_{n\geq 1} | S_{\nu\wedge n} - a(\nu \wedge n) | \|_\Phi \leq \rho_2 \| V \|_\Phi .$$

This proves the assertion.

**Remarks:**

1) If the necessary and sufficient condition (*) holds then necessarily $\| Y_1 \|_\Phi < +\infty$. In fact, $| Y_1 - a | \leq V$ and so by the monotonity of the norm

$$\| Y_1 - a \|_\Phi \leq \| V \|_\Phi < +\infty.$$

From this by the Minkowsky inequality for norms we get

$$\| Y_1 \|_\Phi \leq \| Y_1 - a \|_\Phi + \| a \|_\Phi = \| Y_1 - a \|_\Phi + | a | \| 1 \|_\Phi < +\infty.$$

2) If $P(K_2 \geq | Y_1 - a | \geq K_1) = 1$ is satisfied with some constants $K_2 > K_1 > 0$, then

$$K_1 \| \nu^{1/2} \|_\Phi \leq \| V \|_\Phi \leq K_2 \| \nu^{1/2} \|_\Phi .$$

Indeed, we have

$$K_1 \nu^{1/2} \leq V \leq K_2 \nu^{1/2}$$

and the monotonity of the norm implies the preceding inequality.

3) The result of Theorem 1 is the generalization of Theorem 1a) of the paper [1] for $p > 1$. In fact, let $\Phi(x) = x^p/p$ with $p > 1$ and $x \geq 0$. Then the necessary and sufficient condition (*) says that the almost sure limit $S_\nu - a\nu$ of the stopped random walk belongs to $H_\Phi = H_p$ if and only if

$$[E((\sum_{i=1}^{\infty}(Y_i - a)^2 \chi(\nu \geq i))^{p/2})]^{1/p} < +\infty$$

holds. In this case

$$q^{-p} c_p E((\sum_{i=1}^{\infty}(Y_i - a)^2 \chi(\nu \geq i))^{p/2}) \leq$$

$$\leq E(| S_\nu - a\nu |^p) \leq E(\sup_{n \geq 1} | S_{\nu \wedge n} - a(\nu \wedge n) |^p)$$

$$\leq C_p E((\sum_{i=1}^{\infty}(Y_i - a)^2 \chi(\nu \geq i))^{p/2})$$

where $c_p > 0$ and $C_p > 0$ are constants depending only on $p$.

From the Davis inequality [4] we also see that the assertion is true also for $p = 1$ as follows:

$$c_1 E((\sum_{i=1}^{\infty}(Y_i - a)^2 \chi(\nu \geq i))^{1/2}) \leq E(\sup_{n \geq 1} | S_{\nu \wedge n} - a(\nu \wedge n) |) \leq$$

$$\leq C_1 E((\sum_{i=1}^{\infty}(Y_i - a)^2 \chi(\nu \geq i))^{1/2})$$

where $c_1, C_1$ are positive constants.

## 3. A Generalized Rosenthal Type Inequality for the Stopped Random Walks

Another more useful inequality can be obtained for $S_\nu - a\nu$ and $\sup_{n\geq 1} | S_{\nu\wedge n} - a(\nu\wedge n) |$ by using a Rosenthal-type inequality. Let $(X_n, F_n), n \geq 0$ be a martingale with $X_0 = 0$ and let $d_0 = 0$, further $d_i = X_i - X_{i-1}, i \geq 1$, be its difference sequence. We introduce the conditional quadratic variation by the formula

$$s = s(X) = (\sum_{i=1}^{\infty} E(d_i^2 \mid F_{i-1}))^{1/2}.$$

Let $\Phi(x)$ be a Young function with finite power $p$. Consider the Young function $\Phi'(x) = \Phi(x^2)$. Then, $\Phi'$ also finite power, namely,

$$p' = \sup_{x>0} \frac{2x^2\phi(x^2)}{\Phi(x^2)} = 2p.$$

Let us also suppose that the conjugate Young function $\Psi'(x)$ of $\Phi'$ has also finite power $q'$. Then Theorem 1 and Theorem 2 of [2] say that

$$(q')^{-1} c_\Phi [\sum_{i=1}^{\infty} E(\Phi(d_i^2)) + E(\Phi(s^2))] \leq$$

$$\leq E(\Phi((\lim_{n\to+\infty} \text{ a.s. } X_n^2))) \leq E(\Phi(X^{*2})) \leq$$

$$\leq C_\Phi [\sum_{i=1}^{\infty} E(\Phi(d_i^2)) + E(\Phi(s^2))],$$

where $c_\Phi > 0$ and $C_\Phi > 0$ are constants depending only on $\Phi$. This is a Rosenthal inequality. In the language of the Young function $\Phi'$ this can be expressed in the following form:

$$(q')^{-1}c_\Phi[\sum_{i=1}^{\infty} E(\Phi'(|\, d_i \,|)) + E(\Phi'(s))] \leq$$

$$\leq E(\Phi'(\lim_{n\to+\infty} \text{ a.s. } |\, X_n \,|)) \leq E(\Phi'(X^*)) \leq$$

$$\leq C_\Phi[\sum_{i=1}^{\infty} E(\Phi'(|\, d_i \,|)) + E(\Phi'(s))].$$

The quantities in this inequality are quite simple when considering the stopped random walks. We formulate the corresponding result in the language of the preceding inequality in the Orlicz-spaces.

**Theorem 2.** Let us suppose that the Young function $\Phi$ has finite power $p$. Consider the Young function $\Phi'(x) = \Phi(x^2)$ and its conjugate $\Psi'(x)$. Suppose that the power of $\Psi'(x)$ is also finite and denote it by $q'$. Then, the almost sure limit of the stopped martingale belongs to $H_\Phi$, if and only if $E(\Phi'(|\, Y_1 \,|))$ and $E(\Phi'(\nu^{1/2}))$ are finite. In this case we have the inequality

$$(q')^{-1}c_\Phi \min(1,\sigma^{2p})[E(\Phi'(|\, Y_1 - a \,|))E(\nu) + E(\Phi'(\nu^{1/2}))] \leq$$

$$\leq E(\Phi'(|\, S_\nu - a\nu \,|)) \leq E(\Phi'(\sup_{n\geq 1} |\, S_{\nu\wedge n} - a(\nu \wedge n) \,|)) \leq$$

$$\leq C_\Phi \max(1,\sigma^{2p})[E(\Phi'(|\, Y_1 - a \,|))E(\nu) + E(\Phi'(\nu^{1/2}))],$$

where $c_\Phi > 0$ and $C_\Phi > 0$ are constants depending only on $\Phi$.

**Proof.** The power of $\Phi'$ is now equal to $2p$. The quantities in the above Rosenthal-type inequality are the following:

$$\sum_{i=1}^{\infty} E(\Phi'(|\, d_i \,|)) = \sum_{i=1}^{\infty} E(\Phi(d_i^2)) = \sum_{i=1}^{\infty} E(\Phi((Y_i - a)^2\chi(\nu \geq i))) =$$

$$= \sum_{i=1}^{\infty} E(\Phi((Y_i - a)^2))P(\nu \geq i) = E(\Phi((Y_1 - a)^2))E(\nu) =$$

$$= E(\Phi'(|\, Y_1 - a \,|))E(\nu)$$

and

$$E(\Phi'(s)) = E(\Phi(s^2)) = E(\Phi(\sum_{i=1}^{\infty} E((Y_i - a)^2 \mid F_{i-1})\chi(\nu \geq i))) =$$

$$= E(\Phi(\sigma^2\nu)) = E(\Phi'(\sigma\nu^{1/2})),$$

where we have denoted $\sigma^2 = \text{Var}\, Y_1$. Now the inequality is the following:

$$(q')^{-1}c_\Phi[E(\Phi'(\mid Y_1 - a \mid))E(\nu) + E(\Phi'(\sigma\nu^{1/2}))] \leq$$

$$\leq E(\Phi'(\mid S_\nu - a\nu \mid)) \leq E(\Phi'(\sup_{n\geq 1} \mid S_{\nu\wedge n} - a(\nu \wedge n) \mid)) \leq$$

$$\leq C_\Phi[E(\Phi'(\mid Y_1 - a \mid))E(\nu) + E(\Phi'(\sigma\nu^{1/2}))].$$

The left-hand the right-hand sides are finite if so are $\sigma^2$ and $E(\nu)$. In fact, from the finiteness of $E(\Phi'(\mid Y_1 - a \mid))$ the finiteness of $\sigma^2$ follows. Also, from the finiteness of $E(\Phi(\nu^{1/2}))$ we can deduce the finiteness of $E(\nu)$. For this purpose let us remark that the right-hand side derivative of $\Phi'(x)$ is $2x\phi(x^2)$. Let $x_0 > 0$ be such a quantity for which $2x_0\phi(x_0^2) > 0$. Then trivially

$$\Phi'(x) \geq 2x_0\phi(x_0^2)(x - x_0)^+, \quad x\epsilon[0, +\infty).$$

This implies that

$$E(\Phi'(\nu^{1/2})) \geq 2x_0\phi(x_0)E((\nu - x_0)^+).$$

Consequently,

$$E(\nu) \leq E((\nu - x_0)^+) + x_0 \leq \frac{E(\Phi'(\nu^{1/2}))}{2x_0\phi(x_0^2)} + x_0.$$

Therefore, $E(\Phi'(\nu^{1/2})) < +\infty$ implies that $E(\nu) < +\infty$. Similarly we can prove that from $E(\Phi'(\mid Y_1 - a \mid)) = E(\Phi((Y_1 - a)^2)) < +\infty$ it follows that $\sigma^2 = E((Y_1 - a)^2)$ is finite.

By the convexity of $\Phi'$ and by the fact that $\Phi'$ has finite power $2p$, we have

$$\min(1, \sigma^{2p})E(\Phi'(\nu^{1/2})) \leq E(\Phi'(\sigma\nu^{1/2})) \leq$$
$$\leq \max(1, \sigma^{2p})E(\Phi'(\nu^{1/2})).$$

Consequently, our inequality for the stopped random walks is the following:

$$(q')^{-1}c_\Phi \min(1,\sigma^{2p})[E(\Phi'(|\, Y_1 - a \,|))E(\nu) + E(\Phi'(\nu^{1/2}))] \leq$$

$$\leq E(\{\Phi'(|\, S_\nu - a\nu \,|)) \leq E(\Phi'(\sup_{n\geq 1} |\, S_{\nu\wedge n} - a(\nu \wedge n) \,|)) \leq$$

$$\leq C_\Phi \max(1,\sigma^{2p})[E(\Phi'(|\, Y_1 - a \,|))E(\nu) + E(\Phi'(\nu^{1/2}))].$$

Thus for $S_\nu - a\nu \epsilon H_\Phi$, it is necessary and sufficient that $E(\Phi'(|\, Y_1 - a \,|))$ and $E(\Phi'(\nu^{1/2}))$ be finite.

This proves the assertion.

**Remark.** In [1] the same inequality is proved for the case of $\Phi(x) = x^p/p$ with $p \geq 2$. So, the assertion of the present theorem is the generalization of this special case.

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