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ON AN INEQUALITY OF M.J. KLASS

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Abstract. Let $Y_1, Y_2, \ldots$ be a sequence of i.i.d. zero-mean random variables with the partial sums $S_n = Y_1 + \ldots + Y_n, n \geq 1$. Let ν be any (possibly randomized) stopping time with respect to $\{Y_n\}$. Let further $a_n = E(|S_n|)$ and suppose ν is independent of $\{Y_n\}$. If the Wald equation $E(S_\nu) = E(Y_1)E(\nu) = 0E(\nu) = 0$ holds then technically we require $E(|S_\nu|) < +\infty$ and in this case $E(a_\nu) < +\infty$, since $E(|S_\nu|) = E(\sum_{i=1}^{\infty} |S_i| \chi(\nu = i))$ and ν and $\{Y_n\}$ are independent. Hence, to obtain $E(S_\nu) = 0$ for all stopping times having common marginal distribution, $E(a_\nu) < +\infty$ is a minimal necessary condition on that distribution. M.J. Klass in [1] has proved that this condition is also sufficient. Namely, he proved the following interesting inequality: for a power $p \geq 1$ we have

$$E(\sup_{n \geq 1} |S_{\nu \wedge n}|^p) \leq CE(a_\nu^*),$$

which by the uniform integrability implies the validity of Wald's equation. Here $a_n^* = E(\max_{1 \leq i \leq n} |S_i|^p)$ and $C > 0$ is a constant depending only on p. The aim of the present note is to sharpen this result and to prove the following two-sided inequality: for $p \geq 1$,

$$cE(a_\nu^{(p)}) \leq E(\sup_{n \geq 1} |S_{\nu \wedge n}|^p) \leq CE(a_\nu^{(p)}).$$

Here, $a_n^{(p)} = E(|S_n|^p)$ and the constants $c > 0$ and $C > 0$ do not depend on the distribution of ν. In such a way our two-sided inequality is an improvement of M.J. Klass' one in L^p-spaces.

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1. Introduction and Summary

Let $Y_1, Y_2, \ldots$ be a sequence of independent and identically distributed random variables (i.i.d.) and consider the generalized random walk defined by $S_0 = 0$, $S_n = Y_1 + \ldots + Y_n, n \geq 1$. In this paper we suppose that $E(Y_1) = 0$. Let ν be any (possibly randomized) stopping time with respect to the increasing sequence of σ-fields $F_n = \sigma(Y_1, \ldots, Y_n), n \geq 1$, such that $P(\nu < +\infty) = 1$. We also consider the stopped random walk $S_0 = 0$ and $\{S_{\nu\wedge n}\}, n \geq 1$, where $S_{\nu\wedge n} = \sum\limits_{i=1}^{n} Y_i \chi(\nu \geq i)$. This stopped random walk has a limit on the event $\{\nu < +\infty\}$, whilst it does not exist on the null event $\{\nu = +\infty\}$ except trivially the case when $P(Y_1 = 0) = 1$. We omit this trivial possibility from our considerations. On the event $\{\nu = +\infty\}$ we define $\lim\limits_{n\to+\infty} S_{\nu\wedge n} = 0$, which, from the point of view of taking expectation does not play any role. Thus on the set Ω of the elementary events we have

$$\lim_{n\to+\infty} S_{\nu\wedge n} = \sum_{i=1}^{\infty} Y_i \chi(+\infty > \nu \geq i) = \sum_{n=1}^{\infty} S_n \chi(\nu = n).$$

We shall denote by S_ν the limit of $S_{\nu\wedge n}$ as $n \to +\infty$ on the event $\{\nu < +\infty\}$. We have

$$S_\nu = \sum_{i=1}^{\infty} Y_i \chi(\nu \geq i) = \sum_{n=1}^{\infty} S_n \chi(\nu = n).$$

The main interest in considering the random variable S_ν is to establish Wald's equation, i.e. to prove under some conditions the validity of the relation

$$E(S_\nu) = E(Y_1)E(\nu) = 0 \cdot E(\nu) = 0.$$

If ν is independent of the sequence $Y_1, Y_2, \ldots$, then $E(| S_\nu |) = \sum\limits_{n=1}^{\infty} E(| S_n |)P(\nu = n)$, or, introducing the notation $a_n = E(| S_n |)$, we have

$$E(| S_\nu |) = \sum_{n=1}^{\infty} E(| S_n |)P(\nu = n) = E(a_\nu).$$

Consequently, if, in addition to the independence, we also suppose that $E(a_\nu) < +\infty$, then $E(S_\nu) = \sum\limits_{n=1}^{\infty} E(S_n)P(\nu = n) = 0$, which is Wald's equation. The idea of M.J. Klass is that for the validity of Wald's equation the finiteness of $E(a_\nu)$ is in the above sense necessary at least when ν and the random variables $Y_1, Y_2, \ldots$ are independent. In his paper [1] M.J. Klass proved that for $p \geq 1$ and without supposing the independence of ν and $Y_1, Y_2, \ldots$, the inequality $E(\sup_{n\geq 1} \mid S_{\nu\wedge n} \mid^p) \leq CE(a_\nu^*)$ holds, where $a_n^* = E(\max_{1\leq j\leq n} \mid S_j \mid^p)$ and $C > 0$ is a constant depending only on p. Now, if $E(a_\nu^*) < +\infty$, then $E(\sup_{n\geq 1} \mid S_{\nu\wedge n} \mid^p) < +\infty$ and this implies already the uniform integrability of $\{S_{\nu\wedge n}\}$ and consequently Wald's equation.

Introduce the notation $a_n^{(p)} = E(\mid S_n \mid^p)$, where $p \geq 1$ is some power. We shall prove the validity of the following two-sided inequality:

$$cE(a_\nu^{(p)}) \leq E(\sup_{n\geq 1} \mid S_{\nu\wedge n} \mid^p) \leq CE(a_\nu^{(p)}),$$

where in the case $1 \leq p \leq 2$ we also suppose that $\sigma^2 = E(Y_1^2) < +\infty$. Here the constants $c > 0$ and $C > 0$ do not depend on the distribution of ν. In such a way we improve and sharpen the inequality of M. Klass.

The idea of the proof will be based on the following known inequalities:

a/ if $p \geq 2$, then

$$c_p[\sigma^p E(\nu^{p/2}) + E(\mid Y_1 \mid^p)E(\nu)] \leq E(\sup_{n\geq 1} \mid S_{\nu\wedge n} \mid^p) \leq$$

$$\leq C_p[\sigma^p E(\nu^{p/2}) + E(\mid Y_1 \mid^p)E(\nu)],$$

where $c_p > 0$ and $C_p > 0$ are constants depending only on p. It is clear that the left- and the right-hand sides are finite if and only if so are $E(\nu^{p/2})$ and $E(\mid Y_1 \mid^p)$. This inequality can be found in [2].

By the monotonicity of the L^p-norms we have

$$\sigma \leq [E(\mid Y_1 \mid^p)]^{1/p}$$

since $p \geq 2$. Further, $\nu \geq 1$ and $p/2 \geq 1$ and so the preceding inequality can be written in the following form:

$$(*) \qquad c_p\sigma^p E(\nu^{p/2}) \leq E(\sup_{n\geq 1} \mid S_{\nu\wedge n} \mid^p) \leq 2C_pE(\mid Y_1 \mid^p)E(\nu^{p/2}).$$

b/ For $1 \leq p \leq 2$ Burkholder and Gundy [3] have proved the following two-sided inequality: if $\sigma^2 = 1$, then

$$(**)\qquad c_{p,d}E(\nu^{p/2}) \le E(\sup_{n\ge1} \mid S_{\nu\wedge n} \mid^p) \le C_p E(\nu^{p/2}),$$

where $C_p > 0$ is a constant depending only on p, whilst $c_{p,d} > 0$ is such a constant which depends not only on p but also on $d = E(\mid Y_1 \mid)$. These authors have given a counterexample proving that on the left-hand side of this inequality one cannot have a universal constant depending only on p.

Employing these inequalities we can thus prove that $E(\nu^{p/2})$ and $E\ (a_\nu^{(p)})$ are equivalent, provided that $E(\mid Y_1 \mid^p) < +\infty$, if $p \ge 2$ and $\sigma^2 < +\infty$, if $1 \le p \le 2$.

2. An Upper Bound for $\mathbf{E}(\sup_{\mathbf{n}\ge\mathbf{1}} \mid \mathbf{S}_{\nu\wedge\mathbf{n}} \mid^{\mathbf{p}})$

In order to prove the right-hand side of our two-sided inequality we need a simple lemma.

Lemma 1. Let $Y_1, Y_2, \dots$ be a sequence of i.i.d. random variables and let $p \ge 2$. Suppose that $E(\mid Y_1 \mid^p) < +\infty$. Then

$$c_p^{(1)}\sigma^p n^{p/2} \le a_n^{(p)} \le C_p^{(1)} E(\mid Y_1 \mid^p) n^{p/2},$$

where $c_p^{(1)} > 0$ and $C_p^{(1)} > 0$ are constants depending only on p.

If $1 \le p \le 2$ and $\sigma^2 = E(Y_1^2)$ is finite then $a_n^{(p)} \le C_p^{(1)}\sigma^p n^{p/2}$.

Proof. Using the Marcinkiewicz-Zygmund inequality([4] and [5]) for $p \ge 1$, we have

$$c_p^{(1)} E((Y_1^2 + \dots + Y_n^2)^{p/2}) \le a_n^{(p)} \le C_p^{(1)} E((Y_1^2 + \dots + Y_n^2)^{p/2}),$$

where $c_p^{(1)} > 0$ and $C_p^{(1)} > 0$ are constants depending only on p. If $p \ge 2$ then by the monotonicity of the L^p-norm we have

$$[E(Y_1^2 + \dots + Y_n^2)]^{1/2} \le [E((Y_1^2 + \dots + Y_n^2)^{p/2})]^{1/p},$$

or, in other form

$$c_p^{(1)}(n\sigma^2)^{p/2} \le c_p^{(1)} E((Y_1^2 + \dots + Y_n^2)^{p/2}) \le a_n^{(p)},$$

which proves the left-hand side of the first inequality.

On the other hand, by the so-called C_r-inequality we have

$$E((Y_1^2+\ldots+Y_n^2)^{p/2}) \leq n^{\frac{p}{2}-1}E(\sum_{i=1}^{n} \mid Y_i \mid^p) = n^{\frac{p}{2}}E(\mid Y_1 \mid^p).$$

Therefore,

$$a_n^{(p)} \leq C_p^{(1)}E((Y_1^2+\ldots+Y_n^2)^{p/2}) \leq C_p^{(1)}E(\mid Y_1 \mid^p)n^{p/2},$$

which proves the right-hand side of the first inequality.

If $1 \leq p \leq 2$, then again by the monotonicity of the L^p-norm we have:

$$[E((Y_1^2+\ldots+Y_n^2)^{p/2})]^{1/p} \leq [E(Y_1^2+\ldots+Y_n^2)]^{1/2} = (n\sigma^2)^{1/2},$$

and so

$$a_n^{(p)} \leq C_p^{(1)}E((Y_1^2+\ldots+Y_n^2)^{p/2}) \leq C_p^{(1)}\sigma^p n^{p/2}.$$

This proves the lemma.

Now, we use Lemma 1 to derive the upper bound for $E(\sup_{n\geq 1} \mid S_{\nu\wedge n} \mid^p)$. In this connection we prove:

Theorem 1. Let $Y_1, Y_2, \ldots$ be a sequence i.i.d. random variables with mean value 0 and for $p \geq 1$ let us denote by $a_n^{(p)}$ the expectation $E(\mid S_n \mid^p)$. Let further ν be an a.s. finite stopping time with respect to the increasing sequence $F_n = \sigma(Y_1, \ldots, Y_n), n \geq 1$, of σ-fields. Then, we have

$$E(\sup_{n\geq 1} \mid S_{\nu\wedge n} \mid^p) \leq CE(a_\nu^{(p)}),$$

where the constant $C > 0$ depends on p and on the distribution of Y_1 and is independent of the choice of ν.

Proof. We can suppose that $E(a_\nu^{(p)}) < +\infty$. First, we prove the assertion for $p \geq 2$. Applying Lemma 1 we have $c_p^{(1)}\sigma^p n^{(p/2)} \leq a_n^{(p)}$ for every $n = 1, 2, \ldots$. From these inequalities we get

$$E(\nu^{p/2}) \leq KE(a_\nu^{(p)}),$$

where $K = \frac{1}{c_p^{(1)}}\sigma^{-p}$ is a positive constant. This implies that $E(\nu^{p/2}) < +\infty$. Also, we have $E(\mid Y_1 \mid^p) < +\infty$, since $E(a_\nu^{(p)}) < +\infty$ and so there exists an index n for which $P(\nu = n) > 0$ and so $E(| S_n |^p) < +\infty$. This, by the submartingale property, implies that $E(| Y_1 |^p) < +\infty$. By using (*) we obtain

$$E(\sup_{n\geq 1} | S_{\nu\wedge n} |^p) \leq CE(a_\nu^{(p)}),$$

where the constant $C = 2C_p \frac{1}{c_p^{(1)}}\sigma^{-p}E(| Y_1 |^p)$ does not depend on choice of ν.

Now, we prove the assertion for $1 \leq p \leq 2$. For this purpose let $b_n^{(p)} = = E((\sum_{i=1}^{n} Y_i^2)^{p/2})$. Since the Y_i's are not equal to 0 with probability 1, the sequence $\{b_n^{(p)}\}$ strictly increases with n and tends to $+\infty$ as $n \to +\infty$. Define $n_0 = 0$ and let $n_k = min\{n : b_n^{(p)} \geq 2^k\}, k = 1, 2, \ldots$. It is clear that $n_0 = 0 \leq n_1 \leq n_2 \leq \ldots$. Using this definition let $\nu^* = sup\{k : n_k \leq \nu\}$. It then follows that $2^{\nu^*} \leq b_{n_{\nu^*}}^{(p)} \leq b_\nu^{(p)}$ and

$$\{\nu^* \geq i\} = \{n_i \leq \nu\} = \{n_{\nu^*} \geq n_i\}.$$

Now by the Burkholder-Davis-Gundy inequality ([6]) we have

$$E(\sup_{n\geq 1} | S_{\nu\wedge n} |^p) \leq C_p E((Y_1^2 + \ldots + Y_\nu^2)^{p/2}).$$

By the definition of ν^* we see that $\nu < n_{\nu^*+1}$. Also, $p/2$ is a concave power. It follows that

$$E((Y_1^2 + \ldots + Y_\nu^2)^{p/2}) \leq E(\sum_{i=0}^{\nu^*}(\sum_{n_i\leq j<n_{i+1}} Y_j^2)^{p/2}) =$$

$$= E\{\sum_{i=0}^{\infty}(\sum_{n_i\leq j<n_{i+1}} Y_j^2)^{p/2}\chi(\nu^* \geq i)\} =$$

$$= E\{\sum_{i=0}^{\infty}(\sum_{n_i\leq j<n_{i+1}} Y_j^2)^{p/2}\chi(\nu \geq n_i)\} = \sum_{i=0}^{\infty} E((\sum_{n_i\leq j<n_{i+1}} Y_j^2)^{p/2})P(\nu \geq n_i).$$

Here we have used the fact that the random variables $(\sum_{n_i\leq j<n_{i+1}} Y_j^2)^{p/2}$ and $\chi(\nu \geq n_i)$ are independent. Consequently, by the definition of the n_i's we have

$$E((\sum_{n_i\leq j<n_{i+1}} Y_j^2)^{p/2}) \leq E((\sum_{j=1}^{n_{i+1}-1} Y_j^2)^{p/2}) < 2^{i+1}.$$

Therefore,

$$E((Y_1^2+\ldots+Y_\nu^2)^{p/2}) \le \sum_{i=0}^{\infty} 2^{i+1} P(\nu \ge n_i) = \sum_{i=0}^{\infty} 2^{i+1} P(\nu^* \ge i) =$$

$$= E(\sum_{i=0}^{\infty} 2^{i+1}\chi(\nu^* \ge i)) = 2E(\frac{2^{\nu*+1}-1}{2-1}) \le 4E(2^{\nu^*}) \le 4E(b_\nu^{(p)}).$$

On the basis of the Marcinkiewicz-Zygmund inequality we see that

$$b_n^{(p)} \le \frac{1}{c_p^{(1)}} E(\mid S_n \mid^p) = \frac{1}{c_p^{(1)}} a_n^{(p)},$$

from which

$$E(b_\nu^{(p)}) \le \frac{1}{c_p^{(1)}} E(a_\nu^{(p)}).$$

Comparing the obtained inequalities we finally obtain for $1 \le p \le 2$ that

$$E(\sup_{n\ge 1} \mid S_{\nu\wedge n} \mid^p) \le C_p E((Y_1^2+\ldots+Y_\nu^2)^{p/2}) \le C_p 4E(b_\nu^{(p)}) \le$$

$$\le 4\frac{C_p}{c_p^{(1)}} E(a_\nu^{(p)}),$$

which was to be proved.

3. A Lower Bound for $\mathbf{E}(\sup_{n\ge 1} \mid \mathbf{S}_{\nu\wedge n} \mid^p)$

Theorem 2. Let $Y_1, Y_2, \ldots$ be a sequence of independent and identically distributed random variables with zero mean. For $p \ge 1$ let us denote by $a_n^{(p)}$ the expectation $E(\mid S_n \mid^p)$. For $1 \le p \le 2$ we suppose that the variance $\sigma^2 = E(Y_1^2)$ is finite and $= 1$. If ν is an a.s. finite stopping time with respect to the increasing sequence $F_n = \sigma(Y_1, \ldots, Y_n)$ $n \ge 1$, of σ-fields, then there exists a constant $c > 0$ which depends on the distribution of Y_1 and on p but not on the choice of ν such that

$$E(\sup_{n\ge 1} \mid S_{\nu\wedge n} \mid^p) \ge cE(a_\nu^{(p)})$$

holds.

Proof. Without the loss of the generality we can suppose that $E(\sup_{n\geq 1} | S_{\nu\wedge n} |^p) < +\infty$. Since $| Y_1 | \leq \sup_{n\geq 1} | S_{\nu\wedge n} |$, it follows from the assumptions that $E(| Y_1 |^p) < +\infty$. We also have supposed that in case $1 \leq p \leq 2$ the variance $\sigma^2 = E(Y_1^2)$ is finite and $= 1$. Thus by Lemma 1 we get for all $p \geq 1$ that $a_n^{(p)} \leq An^{p/2}$ holds with some constant $A > 0$ depending on the distribution of Y_1. Namely, for $p \geq 2$ this is equal to $C_p^{(1)} E(| Y_1 |^p)$, whilst for $1 \leq p \leq 2$ this is $C_p^{(1)}\sigma^p$, where $C_p^{(1)}$ is the constant in the Marcinkiewicz-Zygmund inequality depending only on p. Consequently, $E(a_\nu^{(p)}) \leq AE(\nu^{p/2})$ holds. Applying this to the left-hand side of $(**)$ in case $1 \leq p \leq 2$ and to the left-hand side of $(*)$ in case $p \geq 2$ we finally get

$$c_{p,d}E(a_\nu^{(p)}) \leq c_{p,d}AE(\nu^{p/2}) \leq AE(\sup_{n\geq 1} | S_{\nu\wedge n} |^p),$$

from which

$$\frac{c_{p,d}}{C_p^{(1)}\sigma^p} E(a_\nu^{(p)}) \leq E(\sup_{n\geq 1} | S_{\nu\wedge n} |^p),$$

if $1 \leq p \leq 2$, whilst for $p \geq 2$ we have

$$c_p^{(1)}\sigma^p E(a_\nu^{(p)}) \leq c_p^{(1)}\sigma^p AE(\nu^{p/2}) \leq AE(\sup_{n\geq 1} | S_{\nu\wedge n} |^p),$$

and so

$$\frac{c_p^{(1)}\sigma^p}{C_p^{(1)}E(| Y_1 |^p)} E(a_\nu^{(p)}) \leq E(\sup_{n\geq 1} | S_{\nu\wedge n} |^p).$$

The proof is completed.

The Main Result

If we combine the assertions of Sections 2 and 3 the following result can be formulated:

Theorem 3. Let $Y_1, Y_2, \ldots$ be a sequence of independent and identically distributed random variables with zero mean. Let further ν be an almost surely finite stopping time with respect to the increasing sequence $F_n = \sigma(Y_1, \ldots, Y_n)$ of σ-fields, $n = 1, 2, \ldots$. Let $p \geq 1$ be some power and put

$a_n^{(p)} = E(| S_n |^p), n = 1, 2, \ldots$. In case $1 \leq p \leq 2$ we suppose that the variance $\sigma^2 = E(Y_1^2)$ is finite and $= 1$. Then the inequality

$$cE(a_\nu^{(p)}) \leq E(\sup_{n\geq 1} | S_{\nu\wedge n} |^p) \leq CE(a_\nu^{(p)})$$

holds and on p, where $c > 0$ and $C > 0$ are constants depending only on the distribution of Y_1 and are independent of the choice of ν.

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