Annales Univ. Sci. Budapest., Sect. Comp. 11 (1991) 35-51

APPLICATIONS OF THINNING PROCESSES

V.A. GASANENKO

Dedicated to the memory of Professor J. Mogyoródi

1. Introduction

In the present paper there are obtained some limit theorems for thinning processes on the basis of practical cases.

Let $0 = t_1 < t_2 < ...t_n...$ be a sequence of increasing moments of time. It defines the point process on the time line $t \geq 0$, then any subsequence $t_{i_1} < t_{i_2} < ... < t_{i_n} < ...$ defines thinning flow.

Definition of the thinning flow consists of three parts: initial flow; exclusion rule of parts from the initial flow; flow after using exclusion rule (thinning flow). Authors of papers on thinning processes usually invistigate one from these parts on condition of sufficiency of information about another parts.

A. Rényi (1956) has proposed the following raring scheme of initial reccurent flow: the point belongs to thinning flow with probability p, and the point is excluded with probability $1 - p$. He obtained sufficient conditons of approximation of thinning process by means of Poisson process for $p \to 0$. Necessary and sufficient conditions for this approximation are obtained in the works [6,10].

J.Mogyoródi (1971) has proposed a general raring scheme of initial recurrent flow: let $\tau_i, i \geq 0$ be moments of appearance of events in the recurrent flow and $\{z(i)\}_{i\geq 1}$ be sequence of independent, identically distributed integer-valued random variables, $z(i) \in \{1, 2, ...\}$.

Then $\tau_0 = 0, \tau_{z(1)}, \tau_{z(1)+z(2)}, ..., \tau_{z(1)+z(2)+...+z(k)}, ...$ are moments of appearance of events in the thinning flow.

If $P(z(1) = m) = (1 - p)^{m-1}p$, then it is the scheme of A.Rényi. V.A. Gasanenko (1983) extended Mogyoródi's scheme for general random processes. I.N.Kovalenko (1973) proposed another way for generalization of Rényi's scheme. In section 2 we will propose a simple procedure of calculation of the events' sequence number flow the thinning processes for special cases.

Section 3-5 are devoted to the application of thinning processes for special cases.

2. Basic results and definitions.

Let $\tau_i, i \geq 0$ be moments of appearance of the events in the initial flow and χ_i be indicator of exclusion of the points:

$$\chi_i = \begin{cases} 0, & \text{if } i-\text{th event is excluded;} \\ 1, & \text{if } i-\text{th event is kept.} \end{cases}$$

Then we consider the integer-valued random process

$$\xi(k) = inf\{j \geq 1 : \chi(k+j) = 1\}.$$

It defines the correspondence of sequence numbers to kept events. Let $\beta(k)$ be the number of k-th kept event, then $\beta(k) = \beta(k-1) + \xi(\beta(k-1))$, $k \geq 1$, $\beta(0) = 0$ and moment of appearance of this event in thinning flow is equal to $\tau_{\beta(k)}$. Author [4,5] has obtained limit theorems for thinning processes with the help of process $\xi(k)$. We shall use the following result. Define two sequences of numbers $\beta_v(m), \alpha_v(m)$. If $v(t) \geq 1,\ t = 1, 2, ...$ is integer-valued random process and $P(v(t) = k)\overline{_{t\to\infty}} > P(\eta = k)$, for fixed k=1,2,... where η is random value, $\eta \in 1, 2, ...$, then $\beta_v(m)$ is the uniform speed of convergence to stationary distribution:

$$\beta_v(m) = \sup_{k\geq 1} \mid P(v(m) = k) - P(\eta = k) \mid, \ m \geq 0.$$

If $F_{\leq x}, F_x$ are σ-algebras constructed according to families of random values $\{v(t), t \leq x\}$ and $\{v(t), t = x\}$, then $\alpha(m)$ mixing coefficient.

$$\alpha_v(m) = \sup_{\substack{A\in F_{\leq x} \\ B\in F_{x+m}}} \mid P(AB) - P(A)P(B) \mid, \ m \geq 1.$$

Consider the scheme of series: process $\xi(t)$ depends on the sequence number of series $\xi_n(t), n \geq 1$. The symbol $\overset{W}{\underset{n\to\infty}{\Longrightarrow}}$ denotes the weak convergence of the distribution functions.

Theorem 1. Let $c_n < \infty$ be a sequence, such that for $n \to \infty$ the next conditions hold

$$P(\xi_n(0) < c_n x) \overset{W}{\underset{n\to\infty}{\Longrightarrow}} G_1(x), \quad G_1(0) = 0 \tag{1}$$

$$P(\eta_n < c_n x) \underset{n\to\infty}{\overset{W}{\Longrightarrow}} G_2(x), \tag{2}$$

$$\alpha_{\xi_n}(c_n)c_n^2 \underset{n\to\infty}{\longrightarrow} 0 \tag{3}$$

$$\beta_{\xi_n}(c_n) \underset{n\to\infty}{\longrightarrow} 0 \tag{4}$$

then

$$P(\beta_n(m)c_n^{-1} < x) \underset{n\to\infty}{\overset{W}{\Longrightarrow}} G_1 * G_2^{*(m-1)}(x)$$

here * is symbol of convolution: G_1, G_2 are distribution functions.

Remark 1.

Note that in this case and if $\tau_n * n^{-1} \underset{n\to\infty}{\overset{p}{\longrightarrow}} \mu < \infty$ then it follows the convergence $P(\tau_{\beta n(m)}c_n^1(x) \underset{n\to\infty}{\overset{W}{\Longrightarrow}} G_1 * G_2^{*(m-1)}$ [e.g.1].

3. Flows from GI/G/1/0 systems.

3.1. Flow of customers which are not served.

Definition.

τ_1 is moment of arrival of the i-th customer into system, $\tau_1 = 0$;

e is service time of one customer;

$G(X), F(X)$ are distribution functions of τ_2 and e correspondingly;

$c(k) = P(\tau_k \leq 0 < \tau_{k+1})$ is the probability that the service of the first customer is finished between the arrival into system of k-th and k+1-th ones.

Let the i-th customer have number i and we shall denote by $\beta(k)$ the number of the k-th customer who is not serviced.

Define indicators $\chi(i), i \geq 1$

$$\chi(i) = \begin{cases} 1, & \text{if i-th customer is lost;} \\ 0, & \text{if i-th customer is serviced;} \end{cases}$$

and process $\xi(l) : \xi(l) = \min\{j \geq 1 : \chi(l+j)\}$.

We can obtain the following relation for $\beta(k)$:

$$\beta(k) = \beta(k-1) + \xi(\beta(k-1)).$$

In order to use the theorem 1 we determine the characteristics of process $\xi(l)$. Put $c := c(1)$.

Theorem 2.

$$P(\xi(i) = l) = \begin{cases} C^{l-2}(1-c)q_i, & \text{for } l \geq 2, \\ 1 - q_i, & \text{for } l = 1, \end{cases}$$

$$\lim_{i \to \infty} q_i = (\sum_{k \geq 1} kc(k))^{-1}.$$

Proof. The event $\{\xi(i) = l\}, l = 2$ denotes, that in the interval $[\tau_i, \tau_{i+1})$ one from services is finished and customers arrived at moment τ_{i+k} are serviced till moment τ_{i+k+1}, $k = 1, 2, ..., l-2$. The customer arrived in the moment τ_{i+l-1} is not serviced till moment τ_{i+l}. Denoting by q_i the probability of arrival the i+1-th customer into free system we obtain $P(\xi(i) = l) = q_i c^{l-2}(1-c)$. The first part of theorem 2 is proved.

The arrival of i+1-th customer into the free system coincides with one from the events: the service is finished in the interval $[\tau_k, \tau_{k+1})$ and the customer arrived at moment τ_{k+1} was serviced to some moment x from the interval $\{\tau_i, \tau_{i+1}), k = 0, 1, ..., i-1$. These events are mutually exclusive and so we can write

$$q_i = \sum_{k \geq 1} q_k c(i-k), q_0 = 1, \; c(0) = 0.$$

Then $q_i \geq 0$ is the bounded solution of the discrete renewal equation. Consequently, the second part of theorem 2 is result of well-known theorems of the renewal theory.

We compute now the mixing coefficient.

Let $\Theta(k)$ be the time at which the k-th service is finished. Furthermore,

$$P(\xi(i) = l, \xi(i+n) = m) =$$

$$= \sum_{k=1}^{i} P(\tau_i \leq \Theta(k) < \tau_{k+1}) c^{l-2}(1-c)c^{m-2}\cdot$$

$$\cdot \sum_{1+j_1+\ldots+j_p = n-l-1} c(1+j_1)c(j_2)\ldots c(j_p); \; j_r \geq 1, \; r = \overline{1,q}$$

$$P(\xi(i) = l, \xi(i+n) = m) =$$

$$= \sum_{k=1}^{i} P(\tau_i \leq \Theta(k) < \tau_{i+1}) c^{l-2}(1-c)c^{m-2}\cdot$$

$$\cdot \sum_{j_1+\ldots+j_q = n+i} c(j_1)\ldots c(j_q); \; j_r \geq 1, \; r = \overline{1,q}$$

Estimate the next value

$$J(n) = \left| \sum_{j_1=1}^{n-l-2} \frac{c(j_1+1)}{1-c} \sum_{j_2=1}^{n-l-j_1-1} c(j_2) \sum_{j_{n-l-1}=1}^{n-l-2-(j_1+\ldots+j_{n-l-2})} c(j_{n-l-1}) - \right.$$
$$\left. - \sum_{j_1}^{n+i} c(j_1) \sum_{j_2=1}^{n+i-j_1} c(j_2) \cdot \ldots \cdot \sum_{j_{n-l-1}=1}^{n+i-(j_1+\ldots+j_{n+i-1})} c(j_{n+i}) \right|$$

It may be shown, that

$$J(n) \leq n \sum_{j=r(n)}^{n} c(j) +$$
$$+ \left| \sum_{j_{n-l-1}=1}^{n+i-(j_1+\ldots+j_{n-l-2})} c(j_{n-l-1}) \cdot \ldots \cdot \sum_{j_{n+i}=1}^{n+i-(j_1+\ldots+j_{n+j-1})} c(j_{n+i}) - 1 \right| + \tag{5}$$
$$+ \sum_{j_1=1}^{n} \left| \frac{c(j_1+1)}{1-c} - c(j_1) \right| \stackrel{df}{=} \alpha_1(n) + \alpha_2(n) + \alpha_3(n)$$

The first summand of $\alpha_2(n)$ is the probability of events, that one from services is finished in the interval $[\tau_{i-l}, \tau_{i-l+1})$. For this probability holds the inequality

$$\sum_{k=1}^{l-1} P(\tau_{i-l} \leq \Theta(k) < \tau_{i-l+1}) \geq c^{i-l}.$$

Furthermore

$$\alpha_\xi(n) \leq (1-c)^2 J(n). \tag{6}$$

According to theorem 2 and Kalashnikov's result [9]

$$\beta_\xi(n) \leq H \sum_{k \geq n+1} c(k), \quad H < +\infty. \tag{7}$$

Let function $F(x)$ depend on parameter n on such way, that $c_n \underset{n\to\infty}{\longrightarrow} 1$.

Theorem 3. If the conditions

$$\frac{1-c_n-c_n(2)}{1-c_n} \underset{n\to\infty}{\longrightarrow} 0 \tag{8}$$

$$n^{-1}\tau_n \quad \underset{n\to\infty}{\overset{P}{\longrightarrow}} \quad \mu<\infty \tag{9}$$

hold, then

$$P((1-c_n)\tau_{\beta_{n(m)}}<x) \underset{n\to\infty}{\overset{W}{\Longrightarrow}} (1-exp(-\frac{x}{\mu}))^{*m}.$$

Proof. At first we shall prove the next relation

$$c_n(j+1)=(1-c_n)c_n(j)+o(1-c_n), \quad j\geq 1 \tag{10}$$

For $j=2$ from condition (8) we obtain

$$c_n(2)=(1-c_n)(1-o(1))=(1-c_n)(1-c_n+c_n-o(1))=$$
$$=c_n(1-c_n)+(1-c_n)^2+o(1)(1-c_n)=c_n(1-c_n)+o(1-c_n).$$

Further, let relation (10) be valid for $j=k$. Now we show by induction. Using $1-c_n-c_n(2)=\sum\limits_{k\geq 3}c_n(k)$ and (8)

$$\begin{aligned} c_n(k+1) &= (1-c_n)o(1)-\sum_{l\geq 3, l=k+1} c_n(l)= \\ &= (1-c_n)c_n(k)-(1-c_n)((1-c_n)c_n(k-1)+o(1-c_n)+ \\ &+(1-c_n)o(1)\cdot \sum_{l\geq 3, l=k+1} c_n(l)= \\ &= (1-c_n)c_n(k)+o(1-c_n). \end{aligned}$$

We now examine the conditions of theorem 1. Using the relation (10) as $n\to\infty$

$$\alpha_1([(1-c_n)^{-1}])\to 0, \quad \alpha_3([1-c_n)^{-1}])\to 0.$$

and the convergence $c_n^{i-l} \to 1$ when $i-l < ([(1-c_n)^{-1}])$ leads to $\alpha_2([(1-c_n)^{-1}])\to 0$ as $n\to\infty$.

Further, using (5) $J_n([(1-c_n)]) \to 0$ as $n \to \infty$, according to (6) $(1-c_n)^{-2}\alpha_{\xi_n}([(1-c_n)^{-1}]) \to 0$ and $\beta_\xi([(1-c_n)^{-1}]) \to 0$ according (7).

Finally, we obtain limit of the distribution functions of $\xi_n(1)$ and η_n. According to theorem 2

$$P(\xi_n(1), x) = \sum_{k=1}^{[x]} P(\xi_n(1) = k) = 1 - c_n^{[x]-1},$$

$$P(\eta_n < x) = 1 - mc_n^{[x]-1},$$

where

$$m = \lim_{n\to\infty} q_{[(1-c_n)^{-1}]}(n).$$

From the (10) it follows that $m = 1$. Consequently, applying theorem 1 and remark 1 we obtain

$$P(\beta_n(m)(1-c_n) < x) \to (1 - exp(-x))^{*m}$$

$$P(\tau_{\beta_n(m)}(1-c_n) < x) \to (1 - exp(-\frac{x}{\mu}))^{*m}.$$

Notice, that condition (8) corresponds to condition from the well-known theorem of Belyaev I.K.[2].

3.2. Flow of serviced customers.

Put

$$\sigma(i) = \begin{cases} l, & \text{if customer is serviced,} \\ 0, & \text{if customer is lost;} \end{cases}$$

$$\chi(l) = min\{j \geq 1 : \sigma(j+l) = 1\},$$

$$r(k) = r(k-1) + \chi(r(k-1)).$$

Here $r(k)$ is the sequence number of k-th serviced customer. Definition of $q_j, j \geq 0,\ \ c(i), i \geq 1$ is identical to definition from 3.1.

Theorem 4.

$$P(\chi(l) = i) = c(i) : l \geq 1$$

$$P(\eta = i) \stackrel{df}{=} \lim_{l\to\infty} P(\xi(l) = i) = \sum_{j\geq i} c(j)\Big(\sum_{k\geq 1} kc(k)\Big)^{-1} : \ \ l \geq 1.$$

Proof. The first relation follows from the definition of $c(i)$ and $\chi(l)$. It is not difficult to show $P(\chi(l)=i)=\sum_{j=0}^{l-1} q_j c(l+i-1-j)$. Writing the last expression for $r(l)=o(l)$ as $l\to\infty$

$$\sum_{j=0}^{l-1} q_j c(l+i-1-j)=\sum_{j=0}^{r(l)} q_j c(l+i-1-j)+$$

$$+\sum_{j=r(l)+1}^{l-1}(q_j-q)c(l+i-1-j)+q\sum_{j=r(l)}^{l-1} c(l+i-1-j) \qquad (11)$$

We consider, that the first and second summands of (11) converge to zero. The last summand of (11) converges to $q\sum_{k\geq i} c(k)$.

We now shall prove the limit theorem under conditions $c_n \to 0$ and $\sum_{k\geq 1} c_n(k)k \to\to \infty$.

Theorem 5. If the following conditions hold

$$c_n(k)=(1-c_n)^{k-1}c_n+a_n(k), \qquad (12)$$

$$\sum_{k\geq 1} a_n(k)k \underset{n\to\infty}{\longrightarrow} 0, \qquad (13)$$

$$n^{-1}\tau_n \to \mu<\infty$$

then

$$P(\tau_{r_n^{(m)}} c_n < x) \Longrightarrow (1-exp(-\frac{x}{\mu}))^{*m}.$$

Proof.

To estimate the mixing coefficient we consider the next value

$$J(n)=\sup_l \left|P(x(l)=i,x(l+n)=s)-P(x(l)=i)P(x(l+m)=s)\right|=$$

$$=\sup_l \left|\sum_{j=0}^{l-1} q_j c(l+i-1-j)\sum_{k=0}^{n-i-1} q_k c(n-i+s-k)-\right.$$

$$\left.-\sum_{j=0}^{l-1} q_i c(l+i-1-j)\sum_{k=0}^{l+n-1} q_k c(n+l-1+s-k)\right|=$$

$$= \sup_l \Big|(q_{l+i-1} - \sum_{j=l}^{l+i-1} q_j c(l+i-1+j))(q_{n-i-1+s} -$$

$$- \sum_{k=n-i}^{n-i+s-1} q_k c(n-i-1+s-k)) - (q_{l+i-1} -$$

$$- \sum_{j=l}^{l+i-1} q_j c(l+i-1+j))(q_{l+n-1+s} - \sum_{k=n+l}^{n+l-1+s} q_k c(n+l-1+s-k))\Big|.$$

Then applying (12,13) and the Sheffe theorem we have

$$J([c_n^{-1}]) = o\big(c_n^2 sup((1-c_n)^{c_n^{-1}+l-2+s} - (1-c_n)^{c_n^{-1}+2l-i-2+s})\big).$$

It follows $c_n^{-2}\alpha_{\chi_n}([c_n^{-1}]) \underset{n\to\infty}{\longrightarrow} 0$. According (11-13) $b_{\chi_n}([c_n^{-1}]) \longrightarrow 0$.
It is not difficult to show that

$$P(\chi_n(1) < c_n^{-1}x) = \sum_{k=1}^{[c_n^{-1}x]} c_n(k) \longrightarrow 1 - e^{-x},$$

$$P(\chi_n < c_n^{-1}x) = \frac{\sum_{k=1}^{[c_n^{-1}x]} kc_n(k) + [c_n^{-1}x] \sum_{k>[c_n^{-1}x]} c_n(k)}{\sum_{k\geq 1} kc_n(k)} \to 1 - e^{-x}.$$

Theorem is proved.

4. Firing of neuron.

In this section we shall consider the interaction of two renewal process.

Let $Z = \{\xi_i,\ i \geq 1\}, H = \{\eta_i, i \geq 1\}$ define two independent reccurent processes and $\tau_i = \sum_{j=1}^{i} \mu_j$ be time of i-th H arrival. We rare H by the next rule: i-th event of H is kept when interval $(\tau_{i-1}, \tau_i]$ has not anyone from events of Z. This scheme of thinning is a matematical model of neuron firing which was proposed in works Colevan R., Gastwirth J.L., Hoopen T., Reuver [3,7]. Put

$$j^+(x) = \sum_{i=1}^{j} \eta_i - x, \text{ if } x \in [\tau_{j-1}, \tau_j),$$

$$\chi(i) = \begin{cases} 1, & \text{if i-th event of H is kept,} \\ 0, & \text{if i-th event of H is lost.} \end{cases}$$

$$\begin{aligned} \xi(l) &= min\{j \geq 1 : \chi(l+j) = 1\}, \\ s &= \sup_{x \geq 0} P(j^+(x) > \eta_1), \\ i &= \inf_{x \geq 0} P(j^+(x) > \eta_1), \\ c &= P(\xi_1 > \eta_1). \end{aligned} \tag{14}$$

$\beta(m) = \beta(m-1) + \xi(\beta(m-1))$ is the sequence number of m-th event of thinning process H' related to H and $\tau_{\beta(m)}$ is moment of appearance of this event in H.

Further, let n be the number of the series and process $Z_n = \{\xi_{i,n}, j \geq 1\}_{n \geq 1}$ depend on n.

Consequently, the process and values $j^+(x)_n$, $\xi_n(l)$, $\beta_n(m)$, s_n, i_n, c_n.

Theorem 6. If the next conditions hold

$$s_n \longrightarrow 0 \tag{15}$$

$$\frac{s_n}{i_n} \longrightarrow 0 \tag{16}$$

$$m^{-1}\tau_m \underset{m \to \infty}{\longrightarrow} \mu \tag{17}$$

then

$$P(\tau_{\beta_n(m)} c_n < x) \underset{n \to \infty}{\overset{W}{\Longrightarrow}} (1 - exp(-\frac{x}{\mu})^{*m}, \ x \geq 0.$$

Proof.

From the construction H' follows the basic equality of this model

$$\begin{aligned} P(\xi(l) = k) = P(j^+(\tau_l) < \eta_{l+1}, j^+(\tau_{l+1} < \eta_{l+2}, \dots, \\ , \dots, j^+(\tau_{l+k-2}) < \eta_{l+k-1}, j^+(\tau_{l+k-1}) > \eta_{l+k}). \end{aligned} \tag{18}$$

Applying (18) and (14) we have

$$\begin{gathered} i_n \leq P(\xi_n(l) = 1) \leq s_n \\ i_n(1 - s_n) \leq P(\xi_n(l) = 2) \leq s_n(1 - i_n) \\ \vdots \\ i_n(1 - s_n)^{k-1} \leq P((\xi_n(l) = k) \leq s_n(1 - i_n)^{k-1}. \end{gathered}$$

Furthermore

$$i_n \frac{1-(1-s_n)^m}{s_n} \le \sum_{k=1}^{m} P(\xi_n(l)=k) \le s_n \frac{1-(1-i_n)^m}{i_n} \tag{19}$$

for $m = 1, 2, \ldots; l \geq 1$.
The next relations are also true

$$\begin{aligned} i_n(1-s_n)^m P(\xi_n(l)=k) \le P(\xi_n(l)=k, \xi_n(l+r)=m) \le \\ \le P(\xi_n(l)=k)s_n(1-i_n)^m; \\ i_n(1-s_n)^m P(\xi_n(l)=k) \le P(\xi_n(l)=k)P(\xi_n(l+r)=m) \le \\ \le P(\xi_n(l)=k)s_n(1-i_n)^m; \end{aligned}$$

$$\text{for } l, r \geq 1; m \geq 1. \tag{20}$$

Combining (19) with (15), (16) we have convergences

$$P(\xi_n(l) < xs_n^{-1}) \underset{n\to\infty}{\overset{W}{\Longrightarrow}} 1 - exp(-x)$$

and

$$P(\xi_n(l) < xi_n^{-1}) \underset{n\to\infty}{\overset{W}{\Longrightarrow}} 1 - exp(-x)$$

for any fixed l.

It is sufficient for the fulfillment of conditions (1), (2), (4) of theorem 1. We now examine condition (3).

According to (20) we have

$$max(\alpha_{\xi_n}([i_n^{-1}]), \alpha_{\xi_n}([s_n^{-1}]) \le s_n \max_{m\ge 0}(s_n(1-i_n)^m - i_n(1-s_n)^m).$$

It is not difficult to obtain that

$$\max_{m\ge 0}(s_n(1-i_n)^m - i_n(1-s_n)^m) \le k(s_n - i_n), k < \infty$$

in other words

$$max(\alpha_{\xi_n}([s_n^{-1}]), \alpha_{\xi_n}([i_n^{-1}])) = o(i_n^2).$$

Then two sequences of constants $[s_n^{-1}], [i_n^{-1}]$ of this model define the same limit distribution function

$$G_m(x) = (1 - exp(\frac{x}{\mu}))^{*m} \text{ for } \tau_{\beta_n(m)}.$$

Noting that $i_n \leq c_n \leq s_n$ we obtain

$$P(\tau_{\beta_n(m)} < c_n^{-1}) \underset{n \to \infty}{\overset{W}{\Longrightarrow}} G_m(x), \quad x \geq 0.$$

Remark 2. For example, conditions (16) are correct if Z_n is Poisson process.

In this section we will consider the scheme which is proposed by Jagers P., Lindwall J. [8].

However, before it to do we have need of the following change of condition (3) from theorem 1.

If r_n is a sequence such that $r_n = o(c_n), r_n \uparrow \infty$ and

$$P(\xi_n(0) < r_n) \underset{n \to \infty}{\overset{P}{\longrightarrow}} > 0 \tag{21}$$

then we can change the mixing coefficient $\alpha_\nu(m)$ for mixing coefficient $\alpha_{\nu,r_m}(m)$.

Let $\nu_{r_n}(t)$ be a truncated process

$$\nu_m(t) = \begin{cases} \mu(t), & \text{if } \nu(t) \leq c_m - r_m, \\ 0, & \text{if } \nu(t) > c_m - r_m : \end{cases}$$

$$F_{\leq x}^{(r_m)} = o(\nu_m(t), t \leq x)$$

then

$$\alpha_{\nu,r_m}(R_m) = \sup_{x \geq 0} \sup_{\substack{A \in F_{\leq x}^{(r_m)} \\ B \in F_{x+c_m}}} | P(AB) - P(A)P(B) | \tag{22}$$

It was proved in the paper [5,p.38] that cange of condition (3) by conditions (21) and (22) $c_n^2 \alpha_{\xi_n, r_n}(c_n) \longrightarrow 0$ as $n \longrightarrow \infty$ keeps the correctness of theorem 1.

Now we describe the scheme from [8].

Let the distance between point-events be defined by the sequence of random values $\{\xi_i\}_{i\geq 1}$. We fix the integer k and number $\epsilon > 0$. We shall mark the point-event i iff $\sum\limits_{l=i}^{i+k} \xi_l \leq \epsilon$.

Now we obtain the limit theorem for this marked point flow as $\epsilon_n \longrightarrow 0$, where n is number of series $n \geq 1$.

Consider, as usually, numbers of marked points.

$$\chi(i) = \begin{cases} l, & \text{if i-th point is marked,} \\ 0, & \text{if i-th point is not marked.} \end{cases}$$

Further

$\eta(l) = min\{j \geq 0 : \chi(l+j) = 1\}$,
$\beta(m) = \beta(m-1) + \eta(\beta(m-1)), m \geq 1$,
$\beta(0) = 0$.

We will assume that sequence $\{\xi_i\}_{i\geq 1}$ is stationary in strict sense. Then

$$\sigma(t-s) = \sup_{\substack{A\in F_{\leq s} \\ B\in F_{\geq t}}} \mid P(AB) - P(A)P(B) \mid, \; t > s$$

is the strong mixing coefficient.

Put $s_i \sum\limits_{l=i}^{i+k} \xi_l$, $i \geq 1$; $p_n = P(s_i \leq \epsilon_n)$.

Theorem 7. If the following conditions hold

$$p_n \longrightarrow 0, \text{ as } \epsilon \longrightarrow 0 \tag{23}$$

$$\exists a_n : \delta(a_n) = o(p_n^2) \;\&\; a_n = o(p_n^{-1}) \tag{24}$$

$$\lim_{n\to\infty} k_n \sum_{j}^{l_n - 1} X(l_n - j) P(s_1 \leq \epsilon_n, s_{1+j} \leq \epsilon_n) = 0 \tag{25}$$

where k_n and l_n are arbitrary fixed integer-valued functions connected with the next relations

$$\begin{gathered} k_n \longrightarrow \infty, \; l_n \longrightarrow \infty \\ p_n l_n k_n \longrightarrow 1 \end{gathered} \tag{26}$$

$$k_n \delta((p_n^1 - k_n l_n) k_n^{-1}) \longrightarrow 0$$

then

$$P(\beta_n(m)p_n < x) \underset{n\to\infty}{\overset{W}{\Longrightarrow}} (1-exp(-x))^{*m}, x \geq 0.$$

Note, that according to Welsh R.E. [14] if $\delta(t)$ converenges to zero monotonely then condition (26) has solution.

Proof. Definition leads to

$$P(\eta(l) = m) = P(\eta(1) = m) = P(s_1 > \epsilon, ..., s_{m-1} > \epsilon, s_m \leq \epsilon),$$

$$P(\eta(1) < m) = 1 - P(\eta(1) \geq m+1) = 1 - P(s_1 > \epsilon, ..., s_m \leq \epsilon).$$

According to strict stationarity of $\{\xi_i\}$ condition (4) is satisfied and the expressions of limit functions (1), (2) coincide.

Using the equality $P(CB) = P(C) - P(C\bar{B})$ and putting

$$\{s_i > \epsilon_n\} = A_i,\ \{s_i \leq \epsilon_n\} = \bar{A}_i,\ b_n = [(p_n^{-1} - k_n l_n)k_n^{-1}]$$

we obtain

$$P(\eta_n(1) > p_n^{-1}) = P(\bigcap_{i=1}^{[p_n^{-1}]} A_i) =$$
$$= P(A_1 \cdot ... \cdot A_{l_n} \cdot A_{l_n+b_n+1} \cdot ... \cdot A_{2l_n+b_n} \cdot ... \cdot A_{(k_n-1)(l_n+b_n)+1} \cdot$$
$$... \cdot A_{k_n l_n + (k_n-1)b_n})) - P(\bigcap_{j=0}^{k_n-1} (A_{j(l_n+b_n)+1} \cdot ... \cdot A_{(j+1)l_n+jb_n}) \cap$$
$$(\bigcup_{j=0}^{k_n-1} (\bar{A}_{j(l_n+(j-1)b_n+1)} \cdot ... \cdot A_{j(l_n+b_n)}))). \tag{27}$$

According to [14, p.242], the first summand of (27) has estimate

$$|P(\bigcap_{j=0}^{k_n-1} (A_{j(l_n+b_n)+1} \cdot ... \cdot A_{(j+1)l_n+jb_n}))-$$
$$- \prod_{j=0}^{k_n-1} P(A_{j(l_n+b_n)+1} \cdot ... \cdot A_{(j+1)l_n+jb_n})| \leq \delta(b_n)k_n.$$

Furthermore

$$P(A_1 \cdot ... \cdot A_{l_n}) = 1 - P(\bigcup_{j=1}^{l_n} \bar{A}_j),$$

according to Bonferon's inequality we have

$$c_1 - c_2 \leq P(\bigcup_{j=1}^{l_n} \bar{A}_j) \leq c_1$$

where

$$c_1 = \sum_{j=1}^{l_n} P(\bar{A}_j), \quad c_2 = \sum_{1\le i\le j\le l_n} P(\bar{A}_i\bar{A}_j) = \sum_{j=1}^{l_n-1}(l_n - j)P(\bar{A}_1\bar{A}_{1+j}).$$

It leads to

$$\prod_{j=0}^{k_n-1} P(A_{j(l_n+b_n)+1}\cdot\ldots\cdot A_{(j+1)l_n+jb_n}) = (1 - l_np_n + \Delta_n)^{k_n} =$$

$$= (1 - l_np_n)^n + \sum_n$$

where

$$\sum_n \le \sum_{1\le m\le k_n-1} \frac{(k_nc_2)^m}{m!} \le exp(k_nc_2) - 1,$$

Consequently, according (25), (26) it converges to $exp(-1)$. The second summand of (27) is not more than $k_nb_np_n = k_n[(p_n^{-1} - k_nl_n)k_n^{-1}]p_n$ and it converges to zero according (26). Changing p_n^{-1} by xp_n^{-1} and k_n by xk_n for fixed $x \ge 0$ we obtain

$$P(\eta_n(1) < xp_n^{-1}) \longrightarrow 1 - exp(-x).$$

Finally, we examine condition (22). Note, that condition (3) in this case is not valid.

Define the sequence r_n from (21). This sequence is determined from condition

$$P(\eta_n(1) \le r_n) \underset{n\to\infty}{\longrightarrow} 0 \qquad (28)$$

Furthermore

$$P(\eta_n(1) \le r_n) = \sum_{i=1}^{r_n} P(\eta_n(1) = i) \le r_np_n,$$

and, clearly, if $r_n = 0(p_n^{-1})$ then (21) is valid.

According to definitions

$$\alpha_{\eta_n,r_n}([p_n^{-1}]) \le \delta(r_n).$$

Putting $r_n = \alpha_n$ and using condition (24), we obtain $r_np_n \longrightarrow 0, p_n^{-2}\delta(r_n) \longrightarrow 0)$.

Proof is completed.

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VITALIJ GASANENKO
Institute of Mathematics,
Ukrainian Academy of Sciences
252601 Kiev, Repina st. 3
USSR

Received: December 20, 1990.