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ON STABLE AND MIXING SEQUENCES OF σ-FIELDS

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This is the last common work that we have done with Prof. Mogyoródi before his sudden death in Holland as he was visiting the Univ. of Amsterdam in March 1990. A typed manuscript was forwarded from Holland to Dr. Tamás Móri to whom we are grateful for revising it.

1. Introduction

A. Rényi in [1] and [2] has introduced the notion of mixing and stable sequences of events. Let $(\Omega, \mathcal{A}, P)$ be a probability space and let $\{B_n\}$ be a sequence of events. We say that $\{B_n\}$ is *mixing with density* d, where d is a constant, $0 < d < 1$, if for all $A \in \mathcal{A}$ we have

$$\lim_{n\to\infty} P(A \cap B_n) = dP(A).$$

The sequence $\{B_n\}$ is called *stable*, if for all $A \in \mathcal{A}$

$$\lim_{n\to\infty} P(A \cap B_n) = Q(A),$$

where Q is a measure. The stability of a sequence $\{B_n\}$ of events is a more general notion, since in case of mixing sequences of events we have $Q(A) = dP(A)$, where $0 < d < 1$ is a constant. A. Rényi proved that for the stability of $\{B_n\}$ it is necessary and sufficient that $\{\chi_{B_n}\}$ converge weakly in the L_2-sense to a random variable β. (Cf. [2].)

Let $\{B_n\}$ be a mixing sequence of events with density d. Then the sequence $\overline{B_n} = \Omega / B_n$ is also mixing, with density $1 - d$. In fact,

$$\lim_{n\to\infty} P(A \cap \overline{B_n}) = P(A) - \lim_{n\to\infty} P(A \cap B_n) = (1-d)P(A).$$

Since $0 < d < 1$, we see that for arbitrary $A \in \mathcal{A}$,

$$\lim_{n\to\infty} P(A \mid B_n) = \lim_{n\to\infty} \frac{P(A \cap B_n)}{P(B_n)} = \frac{dP(A)}{d} = P(A)$$

and

$$\lim_{n\to\infty} P(A \mid \overline{B_n}) = \lim_{n\to\infty} \frac{P(A \cap \overline{B_n})}{P(\overline{B_n})} = \frac{(1-d)P(A)}{(1-d)} = P(A).$$

Denoting by $\mathcal{Y}_n$ the σ-field $(\emptyset, \Omega, B_n, \overline{B_n})$, it follows from these relations that the conditional probabilities

$$P(A \mid \mathcal{Y}_n) = P(A \mid B_n)\chi_{B_n} + P(A \mid \overline{B_n})\chi_{\overline{B_n}} \quad n = 1, 2, \ldots,$$

converge a.s. to $P(A)$ as $n \to \infty$.

Let $\{B_n\}$ be a stable sequence of events and suppose that $0 < Q(\Omega) < 1$. Then it is easily seen that

$$\lim_{n\to\infty} P(A \mid B_n) = \frac{Q(A)}{Q(\Omega)}$$

and

$$\lim_{n\to\infty} P(A \mid \overline{B_n}) = \frac{P(A) - Q(A)}{1 - Q(\Omega)}.$$

Denoting again by $\mathcal{Y}_n$ the σ-field $(\emptyset, \Omega, B_n, \overline{B_n})$ we see that

$$P(A \mid \mathcal{Y}_n) = P(A \mid B_n)\chi_{B_n} + P(A \mid \overline{B_n})\chi_{\overline{B_n}}$$

converges in L_2-weak sense to a random variable if and only if this holds for the variables χ_{B_n}. In this case the L_2-weak limit of $P(A \mid \mathcal{Y}_n)$ is

$$\frac{Q(A)}{Q(\Omega)}\beta + \frac{P(A) - Q(A)}{1 - Q(A)}(1 - \beta),$$

where β denotes the L_2-weak limit of χ_{B_n}.

In case of mixing sequences of events the a.s. limit of $P(A \mid \mathcal{Y}_n)$ is at the same time an L_2-weak limit. Consequently, a possible generalization of the two notions can be given as follows:

Definition 1. Let $\{\mathcal{Y}_n\}$ be a sequence of σ-fields. Then $\{\mathcal{Y}_n\}$ is called *stable* if for arbitrary $A \in \mathcal{A}$ the L_2-weak limit β_A of the sequence $\{P(A \mid \mathcal{Y}_n)\}$ exists, i.e. if for every $X \in L_2 = L_2(\Omega, \mathcal{A}, P)$ we have

$$\lim_{n\to\infty}\int_\Omega P(A \mid \mathcal{Y}_n)X dp = \int_\Omega \beta_A X dP.$$

Further, the sequence $\{\mathcal{Y}_n\}$ is called *mixing*, if the L_2-weak limit of $\{P(A \mid \mathcal{Y}_n)\}$ is equal to $P(A)$ a.s.

The following simple properties of the L_2-weak limit β_A will often be used in the sequel.

Lemma 1.

(i) β_A is unique a.s. and $\qquad E\beta_A = P(A)$.
(ii) $\beta_\emptyset = 0, \beta_\Omega = 1$ a.s.
(iii) If $A \subset B$, or more generally, $P(A - B) = 0$, then $\beta_A \le \beta_B$ a.s.
(iv) For an arbitrary sequence of events $A_1, A_2, \ldots \beta_{\cup A_i} \le \sum \beta_{A_i}$ a.s. and the a.s. equality holds iff $P(A_i \cap A_j) = 0$ for $i \neq j$.

Proof. The proof of (i) and (ii), being trivial, is left to the reader.

(iii) Let $X = \chi(\beta_A > \beta_B)$, then

$$0 \ge E[(\beta_B - \beta_A)X] = \lim_{n\to\infty} E[P(B \mid \mathcal{G}_n)X - P(A \mid \mathcal{G}_n)X] \ge 0,$$

hence $X = 0$ a.s.

(iv) Suppose first that $A_1, A_2, \ldots, A_N$ are pairwise disjoint events. Then for every $X \in L_2$

$$E\Big[P(\bigcup_{i=1}^N A_i \mid \mathcal{G}_n)X\Big] = E\Big[\sum_{i=1}^N P(A_i \mid \mathcal{G}_n)X\Big] =$$

$$= \sum_{i=1}^N E\Big[P(A_i \mid \mathcal{G}_n)X\Big].$$

Taking limit as $n \to \infty$ we obtain that $\beta_{\bigcup_{i=1}^N A_i} = \sum_{i=1}^N \beta_{A_i}$. Now, let $A_1, A_2, \ldots$ be an infinite sequence of disjoint events. Then, by (iii), $\beta_{\cup A_i} \ge \sum_i \beta_{A_i}$. Since the expectations of the two sides coincide, a.s. equality must hold. This easily implies for arbitrary events that $\beta_{\bigcup\limits_i A_i} \le \sum\limits_i \beta_{A_i}$. Finally, a.s. equality holds iff $E(\beta_{\cup A_i}) = P(\bigcup\limits_i A_i)$ is equal to $E(\sum\limits_i \beta_i) = \sum\limits_i P(A_i)$.

2. Characterizations of mixing.

Theorem 1. $\{\mathcal{G}_n\}$ is a mixing sequence of σ-fields iff for all $A \in \mathcal{A}$ we have

$$Var(P(A \mid \mathcal{G}_n)) \to 0 \qquad n \to \infty,$$

i.e. iff $\{P(A \mid \mathcal{G}_n)\}$ converges in L_2 to $P(A) = E(P(A \mid \mathcal{G}_n))$.

Proof. If $\{\mathcal{G}_n\}$ is mixing then the L_2-weak limit of $\{P(A \mid \mathcal{G}_n)\}$ is $P(A)$ a.s. Consequently, with $X = \chi_A$ we have

$$\lim_{n\to\infty} \int\limits_\Omega P(A \mid \mathcal{G}_n)\chi_A dP = \int\limits_\Omega P(A)\chi_A dP = P^2(A).$$

On the other hand

$$\int\limits_\Omega P(A \mid \mathcal{G}_n)\chi_A dP = \int\limits_\Omega P(A \mid \mathcal{G}_n)P(A \mid \mathcal{G}_n) dP =$$

$$= E(P^2(A \mid \mathcal{G}_n)).$$

This, compared to the preceding limit relation, then gives

$$\lim_{n\to\infty} E(P^2(A \mid \mathcal{G}_n)) = P^2(A).$$

Therefore,

$$\lim_{n\to\infty} Var(P(A \mid \mathcal{G}_n)) =$$

$$= \lim_{n\to\infty} E(P^2(A \mid \mathcal{G}_n)) - \lim_{n\to\infty} E^2((A \mid \mathcal{G}_n)) = P^2(A) - P^2(A) = 0$$

In order to complete the proof we only have to notice that L_2-convergence implies L_2-weak convergence.

Another equivalent formulation is given by:

Corollary 1. $\{\mathcal{G}_n\}$ is mixing sequence of σ-fields if and only if for all $A \in \mathcal{A}$ the sequence $\{P(A \mid \mathcal{G}_n)\}$ of conditional probabilities converges to $P(A)$ in probability.

Proof. If $\{\mathcal{G}_n\}$ is mixing then on the basis on Theorem 1 the sequence $\{P(A \mid \mathcal{G}_n)\}$ converges to $P(A)$ in L_2 and, consequently, in probability too. Conversely, the convergence in probability of $\{P(A \mid \mathcal{G}_n)\}$ to $P(A)$ implies the convergence in L_p for $1 \leq p < \infty$, since $\mid P(A \mid \mathcal{G}_n) - P(A) \mid \leq 1$ a.s., and one can use the dominated convergence theorem. Therefore $\{\mathcal{G}_n\}$ is mixing.

This shows that our present notion of mixing σ-fields coincides with that one introduced in [3].

A third characterization of mixing is the following:

Corollary 2. A sequence $\{\mathcal{G}_n\}$ of σ-fields is mixing iff for all $A \in \mathcal{A}$

$$\sup_{B \in \mathcal{G}_n} \mid P(A \cap B) - P(A)P(B) \mid \to 0, \quad n \to \infty.$$

Proof. Suppose $\{\mathcal{G}_n\}$ is mixing. Then $\{P(A \mid \mathcal{G}_n)\}$ converges to $P(A)$ in probability. Therefore

$$\sup_{B \in \mathcal{G}_n} \mid P(A \cap B) - P(A)P(B) \mid = \sup_{B \in \mathcal{G}_n} \mid \int_B (P(A \mid \mathcal{G}_n) - P(A))dP \mid \leq$$

$$\leq \sup_{B \in \mathcal{G}_n} \int_B \mid P(A \mid \mathcal{G}_n) - P(A) \mid dP \leq \int_\Omega \mid P(A \mid \mathcal{G}_n) - P(A) \mid dP$$

and the last term in this inequality tends to 0 by the dominated convergence theorem as $1 \geq \mid P(A \mid \mathcal{G}_n) - P(A) \mid \to 0$ in probability.

Conversely, suppose that

$$\sup_{B \in \mathcal{G}_n} \mid P(A \cap B) - P(A)P(B) \mid \to 0, \quad n \to \infty.$$

Let $B = B_1 \cup B_2 = \{P(A \mid \mathcal{G}_n) - P(A) \geq \epsilon\} \cup \{P(A \mid \mathcal{G}_n) - P(A) \leq -\epsilon\}$. Then $B_1 \in \mathcal{G}_n$ and $B_2 \in \mathcal{G}_n$. Further

$$\int_{B_1} (P(A \mid \mathcal{G}_n) - P(A))dP \geq \epsilon P(B_1)$$

and

$$\int_{B_2} (P(A \mid \mathcal{G}_n) - P(A))dP \leq -\epsilon P(B_2)$$

From these relations it follows that

$$\left|\int_{B_1} (P(A \mid \mathcal{G}_n) - P(A))dP\right| + \left|\int_{B_2} (P(A \mid \mathcal{G}_n) - P(A))dP\right| \geq$$

$$\geq \epsilon\big(P(B_1) + P(B_2)\big) = \epsilon P(B).$$

So

$$\epsilon P(B) \leq 2 \sup_{B \in \mathcal{G}_n} \mid P(A \cap B) - P(A)P(B) \mid$$

and by assumption the right hand side converges to 0 as $n \to \infty$. From these relations it follows that

$$\lim_{n \to \infty} P\{\mid P(A \mid \mathcal{G}_n) - P(A) \mid \geq \epsilon\} = 0.$$

We have shown that $P(A \mid \mathcal{G}_n)$ converges to $P(A)$ in probability. By Corollary 1 above this implies that the sequence $\{\mathcal{G}_n\}$ is mixing.

This proves the proposition.

3. Stability and mixing.

Let $\{\mathcal{G}_n\}$ be a stable sequence of σ-fields. Denote by β_A the L_2-weak limit of the sequence $\{P(A \mid \mathcal{G}_n)\}$. Consider the family F of the events for which $\beta_A = \chi_A$ a.s. Note that all the trivial events of $\mathcal{A}$ belong to F.

Theorem 2. F is a σ-field, and if $\{\mathcal{G}_n\}$ is mixing then F is trivial.

Proof. $\Omega \in F$, since Ω is a trivial event. If $A \in F$ then $\bar{A} \in F$. In fact $\beta_A = \chi_A$ a.s. and so $\beta_{\bar{A}} = 1 - \beta_A = 1 - \chi_A = \chi_{\bar{A}}$ a.s. Let A_1 and A_2 belong to F. We shall show that $A_1 \cap A_2 \in F$. Observe that $\beta_{A_1 \cap A_2} \leq \chi_{A_i}$ for $i = 1, 2$ implies $\beta_{A_1 \cap A_2} \leq \chi_{A_1 \cap A_2}$. Since $E\beta_A = P(A)$ for all $A \in \mathcal{A}$ we see that

$$\beta_{A_1 \cap A_2} = \chi_{A_1 \cap A_2} \text{ a.s.}$$

This shows that F is an algebra. It remains to prove that for disjoint $A_1, A_2, \ldots \in F$ the union $A = \bigcup_i A_i$ belongs to F, but this follows from the equality $\beta_{\cup A_i} = \sum \beta_{A_i}$.

Assume that $\{\mathcal{G}_n\}$ is mixing,

$$P(A \mid \mathcal{G}_n) \to P(A)$$

in L_2 as $n \to \infty$. By stability for $A \in F$ we have

$$P(A \mid \mathcal{G}_n) \to \chi_A$$

in the L_2-weak sense. From this $P(A) = \chi_A$ a.s. and so A is trivial. Therefore F is trivial.

4. A necessary and sufficient condition for stability and mixing.

Consider the space $L_2 = L_2(\Omega, \mathcal{A}, P)$. This space furnished with the inner product $(X,Y) = EXY$ and norm $\| X \|_2 = \sqrt{EX^2}$ is a Hilbert space.

Let $\{\mathcal{G}_n\}$ be a sequence of sub σ-fields of $\mathcal{A}$. Write $\mathcal{U} = \cup_{n=1}^{\infty} \mathcal{G}_n$. Note that $\mathcal{U} \subset \mathcal{A}$ in general need not be a σ-field. Consider the finite linear combinations of the indicators χ_B with $B \in \mathcal{U}$, and their limits in L_2-norm. The set of these random variables will be denoted by $\mathcal{H}_1$. Then $\mathcal{H}_1$ is a subspace of the Hilbert space L_2. Also consider the subspace $\mathcal{H}_2$ of L_2 which is orthogonal to $\mathcal{H}_1$. The existence of $\mathcal{H}_2$ is guaranteed by the Riesz decomposition theorem. According to this for every $X \in L_2$ we have $X = X_1 + X_2$ with $X_1 \in \mathcal{H}_1, X_2 \in \mathcal{H}_2$ and $EX_1X_2 = 0$.

For the sake of completeness we first prove

Lemma 2. If $X \in \mathcal{H}_2$ then for all n we have

$$E(X \mid \mathcal{G}_n) = 0 \quad \text{a.s.}$$

Proof. Note the $1 \in \mathcal{H}_1$. This implies that $EX = 0$. By definition of $\mathcal{H}_2$ we have $E(\chi_B X) = 0$ for all $B \in \mathcal{G}_n$. It is well known that this gives $E(X \mid \mathcal{G}_n) = 0$ a.s.

We are now in a position to prove

Theorem 3. Let $\{\mathcal{G}_n\}$ be a sequence of sub σ-fields of $\mathcal{A}$. In order that for arbitrary $A \in \mathcal{A}$ the sequence $P(A \mid \mathcal{G}_n)$ converges weakly in L_2 to a limit β_A, it is sufficient that this hold for the sets A in $\mathcal{U} = \cup_{k=1}^{\infty} \mathcal{G}_k$.

Proof. Assume $A \in \mathcal{A}$. Decompose χ_A in the form $\chi_A = X_1 + X_2$, where $X_1 \in \mathcal{H}_1$ and $X_2 \in \mathcal{H}_2$. Then by Lemma 2

$$E(\chi_A \mid \mathcal{G}_n) = P(A \mid \mathcal{G}_n) = E(X_1 \mid \mathcal{G}_n) + 0 \quad \text{a.s.}, n = 1, 2, \ldots$$

Suppose $E(\chi_A \mid \mathcal{G}_n) \to \beta_A$ in the L_2-weak sense for $A \in \mathcal{U}$. By linearity $E(U \mid \mathcal{G}_n) \to \beta_U$ for any random variable U which is a linear combination of such indicator functions, where β_U is the corresponding linear combination of the limit functions β_A. Let $\mathcal{H}_0$ denote the set of linear combinations of indicator functions χ_A with $A \in \mathcal{U}$. Note that $\mathcal{H}_0$ is dense in $\mathcal{H}_1$ by definition of $\mathcal{H}_1$ and that

$$E(U \mid \mathcal{G}_n) \to \beta_U \quad L_2 - \text{weakly for all } U \in \mathcal{H}_0,$$

$$\| \beta_U \|_2 \leq \lim_{n\to\infty} inf \| E(U \mid \mathcal{G}_n) \|_2 \leq \| U \|_2 .$$

We see that $U \to \beta_U$ is a linear contraction from $\mathcal{H}_0$ into L_2. This extends to a linear contraction $V \to \beta_V$ from $\mathcal{H}_1$ into L_2 since L_2 is complete. We have to prove that

$$E(V \mid \mathcal{G}_n) \to \beta_V \quad L_2 - \text{weakly for all } V \in \mathcal{H}_1.$$

Note that

$$E(V \mid \mathcal{G}_n) - \beta_V = E(V - U \mid \mathcal{G}_n) + E(U \mid \mathcal{G}_n) - \beta_U + (\beta_U - \beta_V).$$

Given $X \in L_2$ and $\epsilon > 0$ choose $U \in \mathcal{H}_0$ so that $\| U - V \|_2 < \epsilon/(1 + 2 \| X \|_2)$. Then $|E(E(V - U \mid \mathcal{G}_n)X)| + | EX(\beta_U - \beta_V) | < \epsilon$ and

$$|E(E(V \mid \mathcal{G}_n)X - \beta_V X)| < \epsilon + |E(E(U \mid \mathcal{G}_n)X - \beta_U X)| < 2\epsilon, n \geq n_0.$$

Remark. Under the conditions of Theorem 3 the sequence $P(A \mid \mathcal{G}_n)$ will converge in L_2-weak sense to $P(A)$ if and only if $\beta_A = P(A)$ a.s.

In this connetion we formulate the following:

Theorem 4. (cf. [3]) Let $\{\mathcal{G}_n\}$ be a sequence of sub σ-fields of $\mathcal{A}$. In order that for arbitrary $A \in \mathcal{A}$ the sequence

$$P(A \mid \mathcal{G}_n) \quad n = 1, 2, \ldots$$

converges in probability to $P(A)$, i.e. that $\{\mathcal{G}_n\}$ be a mixing sequence of σ-fields, it is sufficient that this holds for the events B belonging to $\mathcal{U} = \cup_{k=1}^{\infty} \mathcal{G}_k$.

Proof. This is similar to that of the previous assertion:

$$E(U \mid \mathcal{G}_n) \to EU \quad \text{in probability for } n \to \infty$$

for $U \in \mathcal{H}_0$. We have to show that this relation remains valid for elements of $\mathcal{H}_1$.

Let $V \in \mathcal{H}_1$. We may and shall assume that $EV = 0$. Let $\epsilon > 0$. Choose $U \in \mathcal{H}_0$ so that $\| V - U \|_2 < \epsilon^2$. This inequality remains valid if we replace U by $U' = U - EU$. Note that $U' \in \mathcal{H}_0$ and hence $P\{| E(U'|\mathcal{G}_n) | > \epsilon\} < \epsilon$ for $n \geq n_0$. This gives

$$P\{| E(V \mid \mathcal{G}_n) | > 2\epsilon\} \leq P\{| E(U' \mid \mathcal{G}_n) | > \epsilon\} +$$

$$+ P\{| E(V - U' \mid \mathcal{G}_n) | > \epsilon\} \leq \epsilon + \frac{1}{\epsilon^2} E(V - U')^2 < \epsilon + \epsilon^2, \; n \geq n_0.$$

Since ϵ is arbitrary this proves that $E(V \mid \mathcal{G}_n) \to 0$ in probability.

We have the following result:

Theorem 5. Let $\{\mathcal{G}_n\}$ be a stable sequence of σ-fields. Let $Z \in L_2 = L_2(\Omega, \mathcal{A}, P)$ be a square integrable random variable. Then the L_2-weak limit of the sequence $E(Z \mid \mathcal{G}_n)$ exists and

$$E\beta_Z = EZ$$

$$E\beta_Z^2 \leq EZ^2.$$

If, in particular, $\{\mathcal{G}_n\}$ is a mixing sequence of σ-fields, then $\beta_Z = EZ$ a.s. and $E(Z \mid \mathcal{G}_n)$ converges to EZ in L_2.

Proof. L_2-weak convergence was established in the proof of Theorem 3 and implies the stated equality for the expectations (since $E(Z \mid \mathcal{G}_n) = EZ$ and $1 \in L_2$). For the second moment one has the inequality $E\big(E(Z \mid \mathcal{G}_n)^2\big) \leq EZ^2$. This inequality then remains valid for the weak limit: $Y_n \to Y$ weakly in L_2 and $\| Y_n \|_2 \leq 1$ implies $E(Y_n Y) \leq \| Y \|_2$ and hence by weak convergence $EY^2 \leq \| Y \|_2$ which implies $\| Y \|_2 \leq 1$.

Now assume the sequence $\{\mathcal{G}_n\}$ is mixing. Then $E(Z \mid \mathcal{G}_n) \to EZ$ in probability as was shown in the proof of Theorem 4. If Z is bounded this implies convergence in L_2 by Lebesgue's theorem on dominated convergence. Since the bounded variables are dense in L_2, for every $Z \in L_2$ and $\epsilon > 0$ there exists a bounded random variable Z' such that $\| Z - Z' \|_2 < \epsilon$. Now, $\| E(Z \mid \mathcal{G}_n) - E(Z' \mid \mathcal{G}_n) \|_2 < \epsilon$ and $| EZ - EZ' | < \epsilon$. Hence $\| E(Z \mid \mathcal{G}_n) - EZ \|_2 < 2\epsilon + \| E(Z' \mid \mathcal{G}_n) - EZ' \|_2 < 3\epsilon$ for $n \geq n_0$.

Remarks.

1. One can easily verify that the $L_2 \to L_2$ operator $\beta.$ is linear and by the above Theorem 5 it is a contraction, hence continuous. This fact throws new light on Lemma 1 (iv).
2. Let $\{\mathcal{G}_n\}$ be mixing. Then $E(Z \mid \mathcal{G}_n) \to EZ$ in probability for any integrable Z (see [3]).
3. Convergence of $P(A \mid \mathcal{G}_n)$ is determined by the large values of n. Hence we may replace the collection $\mathcal{U}$ in Theorems 3 and 4 above by the smaller collection $\mathcal{U}_n = \bigcup_{k \geq n} \mathcal{G}_k$.

Let $\mathcal{H}_{1n}$ and $\mathcal{H}_{2n}$ be the corresponding complementary subspaces. The sequence $\mathcal{H}_{1n}$ is decreasing and the sequence $\mathcal{H}_{2n}$ is increasing. Let $\mathcal{H}_{1\infty} = \cap \mathcal{H}_{1n}$ and let $\mathcal{H}_{2\infty}$ be the closure of the union $\cup \mathcal{H}_{2n}$. Let $\mathcal{E}$ be the intersection of the σ-fields $\sigma(\mathcal{U}_n)$. Then $\mathcal{H}_{1\infty}$ is a closed linear subspace of $L_2(\Omega, \mathcal{E}, P)$.
Let $X \in L_2(\Omega, \mathcal{A}, P)$ be arbitrary. Then the following equivalences hold:

$$X \perp \mathcal{H}_{2\infty} \Longleftrightarrow X \perp \mathcal{H}_{2n} \text{ for all } n \Longleftrightarrow X \in \mathcal{H}_{1n} \text{ for all } n.$$

Consequently, $\mathcal{H}_{1\infty}$ and $\mathcal{H}_{2\infty}$ are complementary orthogonal subspaces of $L_2\,(\Omega, \mathcal{A}, P)$. Let $X \in \mathcal{H}_{2\infty}$, then one can find a sequence $X_n \in \mathcal{H}_{2n}$ converging to X in L_2. By Lemma 2, $E(X_n \mid \mathcal{G}_n) = 0$ a.s., hence by the contraction property of conditional expectation

$$\| E(X \mid \mathcal{G}_n) \|_2 \leq \| X - X_n \|_2 \to 0.$$

This implies that the sequence $\{\mathcal{G}_n\}$ is mixing if $\mathcal{H}_{1\infty}$ is the one-dimensional subspace of the constant functions. In particular this will be the case if $\mathcal{E}$ is trivial.

5. A consequence and some examples.

Mixing sequences of events in the sense of the definition of A. Rényi now can be constructed in the following way:

Corollary 3. Let $\{\mathcal{G}_n\}$ be a mixing sequence of σ-fields, $B_n \in \mathcal{G}_n, n = 1, 2, \ldots$ a sequence of events such that $\lim_{n \to \infty} P(B_n) = d$ exists. Suppose that $0 < d < 1$. Then $\{B_n\}$ is a mixing sequence of events with density d.

Proof. For arbitrary $A \in \mathcal{A}$ we have

$$| P(A \cap B_n) - dP(A) | \leq \sup_{B \in \mathcal{G}_n} | P(A \cap B) - P(A)P(B) | +$$
$$+ | P(A)P(B_n) - dP(A) | .$$

The first term on the right hand side vanishes by Corollary 2, whilst the second one vanishes by assumption.

Let $\{\mathcal{G}_n\}$ be a decreasing sequence of σ-fields of events. Then by the convergence theorem for the reverse martingales the almost sure limit $\lim_{n\to\infty} P(A \mid \mathcal{G}_n)$ exists and is equal to $P(A \mid \mathcal{G}_\infty)$ where $\mathcal{G}_\infty = \cap_{n=1}^{\infty} \mathcal{G}_n$. $P(A \mid \mathcal{G}_\infty) = \beta_A$ is at the same time the L_2-weak limit of $P(A \mid \mathcal{G}_n)$. Therefore the sequence $\{\mathcal{G}_n\}$ is always stable. It is mixing if and only if $\mathcal{G}_\infty$ is a trivial σ-field.

As a concrete example let $\mathcal{F}_1, \mathcal{F}_2, \ldots$ be a sequence of independent σ-fields and consider the σ-fields

$$\mathcal{B}_n = \sigma(\mathcal{F}_n, \mathcal{F}_{n+1}, \ldots) \quad n = 1, 2, \ldots$$

Then by the Kolmogorov zero-one law the tail σ-field $\mathcal{B} = \cap_{n=1}^{\infty} \mathcal{B}_n$ is trivial. Consequently the conditional probabilities $P(A \mid \mathcal{B}_n)$ converge a.s. and consequently weakly in L_2 to $P(A)$ as $n \to \infty$. This example shows that there exist mixing sequences of σ-fields.

Let $\{\mathcal{G}_n\}$ be an increasing sequence of σ-fields. Then by the martingale convergence theorem for every $A \in \mathcal{A}$ the almost sure limit and consequently the L_2-weak limit $\lim_{n\to\infty} P(A \mid \mathcal{G}_n)$ exists and is equal to $\beta_A = P(A \mid \mathcal{G}_\infty)$, where $\mathcal{G}_\infty = \sigma(\cup_{n=1}^{\infty} \mathcal{G}_n)$. Therefore an increasing sequence $\{\mathcal{G}_n\}$ is always stable and it is mixing if and only if every σ-field $\mathcal{G}_n$ is trivial.

A. Rényi underlined the generality of the condition of stability: If α_n is a bounded sequence in L_2 then we can choose a subsequence which has an L_2-weak limit. Let $C_1, C_2, \ldots$ be events from $\mathcal{A}$. Then $\chi_{C_1}, \chi_{C_2}, \ldots$ is a bounded sequence in L_2. Therefore there exists a subsequence, say χ_{B_n}, which converges in the L_2-weak sense. Let $A \in \mathcal{A}$ be arbitrary. Then

$$P(A \cap B_n) = \int_{\Omega} \chi_{B_n} \chi_A dP \to \int_{\Omega} \beta \chi_A dP = \int_{A} \beta dP, \; n \to \infty$$

where β is the weak limit of χ_{B_n}.

This was the last form J. Mogyoródi was working on before his sudden death. The first two authors decided to preserve the paper in its original form, only making minor changes, paying thus the tribute of respect to the memory of Professor Mogyoródi.

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