Annales Univ. Sci. Budapest., Sect. Comp. 10 (1989) 299-323

A SPLINE APPROXIMATION METHOD FOR THE INITIAL VALUE PROBLEM $y^{(n)} = f(x,y,y')$

THARWAT FAWZY and SAMIA SOLIMAN

Suez-Canal Univ. Ismailia,
Al-Azhar Univ. (Girls)

Abstract: In this paper a method to approximate the solution of the non linear ordinary differential equation of n-th order by spline functions is presented. The existence, uniqueness and convergence of the approximate spline solution are investigated. Moreover the stability of this method is proved. Two numerical examples are considered.

1. Introduction and description of the method.

In this paper we consider the non linear differential equation

$$(1.1) \qquad y^{(n)} = f(x,y,y')$$

where $f \in C([0,1] \cdot R^2)$. We assign to equation (1.1) the initial conditions:

$$(1.2) \qquad y^{(i)}(0) = y_0^{(i)}, \qquad i = 0(1)n-1$$

We assume that $f(x,y,y')$ is a function on R^3 to R defined and continuous in

$$D : | x - x_0 | < \alpha, | y - y_0 | < \beta, | y' - y'_0 | < \gamma.$$

We also assume for $(x, y, y'), (x, y_1, y_1')$ and (x, y_2, y_2') in D:
(1.3)

$$| f(x, y, y') | \leq M$$

and the Lipschitz condition:
(1.4)

$$| f(x, y_1, y_1') - f(x, y_2, y_2') | \leq L(| y_1 - y_2 | + | y_1' - y_2' |)$$

where L is the Lipschitz constant.

We also assume that $w_0(h)$ is the modulus of continuity of $y^{(n)}$ on $[0, 1]$.

Let $y(x)$ be the unique solution of the differential equation (1.1)-(1.2).

Our method is to construct a polynomial spline function of degree $m \leq 2n + 1$ approximating the solution $y(x)$ denoted by $S_\Delta(x)$, where Δ is the mesh points:

$$\Delta : 0 = x_0 < x_1 < \ldots < x_k < x_{k+1} < \ldots < x_N = 1$$

where,

$$x_{k+1} - x_k = h \quad \text{for all} \quad k = 0(1)N - 1$$

If we integrate (1.1) $(n - i)$ times from x_k to x,
where $x_k \leq x \leq x_{k+1}$ and $i = 0(1)n - 1$, putting $x = x_{k+1}$, then we get

$$y_{k+1}^{(i)} = \sum_{j=0}^{n-i-1} \frac{y_k^{i+j}}{j!} h^j + \qquad (1.5)$$

$$+\underbrace{\int_{x_k}^{x_{k+1}} \cdots \int_{x_k}^{t_{n-i-1}}}_{(n-i)\ \text{times}} f(t_{n-i}, y(t_{n-i}), z'(t_{n-i}))dt_{n-i} \dots dt_1$$

The corresponding approximate values are defined as

(1.6)

$$\overline{y}_{k+1}^{(i)} = \sum_{j=0}^{n-i-1} \frac{\overline{y}_k^{i+j}}{j!} h^j +$$

$$+\underbrace{\int_{x_k}^{x_{k+1}} \cdots \int_{x_k}^{t_{n-i-1}}}_{(n-i)\ \text{times}} f(t_{n-i}, y_k^*(t_{n-i}), y_k^{**'}(t_{n-i}))dt_{n-i} \dots dt_1$$

where $i = 0(1)n-1$,

(1.7)

$$y_k^*(t) = \sum_{j=0}^{n} \frac{\overline{y})k^{(j)}}{j!}(t - x_k)^j, x_k \leq t \leq x_{k+1}$$

and

(1.8)

$$y_k^{**'}(t) = \sum_{j=0}^{n-2} \frac{\overline{y}_k^{(j+1)}}{j!}(t - x_k)^j +$$

$$+\underbrace{\int_{x_k}^{t} \cdots \int_{x_k}^{t_{n-2}}}_{(n-1)\ \text{times}} f(t_{n-1}, y_k^*(t_{n-1}), z_k^{*'}(t_{n-1}))dt_{n-1} \dots dt_1$$

and

(1.9)

$$y_k^{*\prime}(t) = \sum_{j=0}^{n-1} \frac{\overline{y}_k^{(j+1)}}{j!}(t - x_k)^j, x_k \leq t \leq x_{k+1}.$$

If $i = n$, then the exact value $\overline{y}_{k+1}^{(n)}$ and the corresponding approximate value $\overline{y}_{k+1}^{(n)}$ are given from the following relations

(1.10)

$$y_{k+1}^{(n)} = f(x_{k+1}, y_{k+1}, y'_{k+1})$$

and

(1.11)

$$\overline{y}_{k+1}^{(n)} = f(x_{k+1}, \overline{y}_{k+1}, \overline{y}'_{k+1})$$

where $\overline{y}_{k+1}$ and $\overline{y}'_{k+1}$ are the approximate values of y_{k+1} and y'_{k+1} respectively.

The Taylor polynomial of the exact solution and its first derivative for $x_k \leq t \leq x_{k+1}$ are given by the relations

(1.12)

$$y(t) = \sum_{j=0}^{n-1} \frac{y_k^{(j)}}{j!}(t - x_k)^j + \frac{y^{(n)}(\xi_k)}{n!}(t - x_k)^n$$

(1.13)

$$y'(t) = \sum_{j=1}^{n-1} \frac{y_k^{(j)}}{(j-1)!}(t - x_k)^{j-1} + \frac{y^{(n)}(n_k)}{(n-1)!}(t - x_k)^{n-1}$$

where $\xi_k, \eta_k \epsilon (x_k, x_{k+1})$

and we shall use the above two expansions in the proof of convergence.

We start the calculation by using the substitutions

$$\overline{y}_0^{(i)} = y_0^{(i)}, \quad i = 0(1)n$$

2. Error Estimations For the Discrete Approximants.

Before proving the main convergence theorems, we give some definitions and introduce a lemma which will help us in arriving at the proof of the theorems.

Definition 2.1.

We denote the estimated errors of $\overline{y}_{k+1}^{(i)}$ at any point $x_{k+1} \in [0,1], i = 0(1)n$, as the following:

$$e_{k+1}^{(i)} = | y_{k+1}^{(i)} - \overline{y}_{k+1}^{(i)} |$$

Lemma 2.1.

The inequality

$$e_{k+1}^{(n)} \leq L(e_{k+1} + e'_{k+1})$$

holds true for all $k = 0(1)N-1$, where L us the Lipschitz constant.

Proof

From the Definition 2.1. and by using equations (1.10) and (1.11) and the Lipschitz condition (1.4), we can get the required result.

Definition 2.2

Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be two matrices of the same order. Then we say that

$$A \leq B$$

iff:

i) a_{ij} and b_{ij} are non negative numbers.

ii) $a_{ij} \leq b_{ij}$ for all i, j.

Theorem 2.1

Let $y_{k+1}^{(i)}$ be the exact values of the solution of (1.1)-(1.2) and its i-th derivatives at $x_{k+1} \in [0,1]$, where $i = 0(1)n-1$. If the corresponding approximate values $\overline{y}_{k+1}^{(i)}$ are given by the formula (1.6), then the error is estimated by the inequality

$$e_{k+1}^{(i)} \leq c_i w_0(h) h^n$$

which holds true for all $k = 0(1)N-1$, where c_i are constants independent of h and $w_0(h)$ is the modulus of continuity of $y^{(n)}(x)$ on $[0,1]$

Proof

By using the Definition 2.1 and by using (1.5), (1.6), the Lipschitz condition (1.4), the expansion (1.12) and (1.13), we get (2.1)

$$e_{k+1}^{(i)} \leq \sum_{j=0}^{n-i-1} \frac{e_k^{i+j}}{j!} h^j + L \sum_{j=0}^{n} \frac{e_k^{(j)}}{(j+n-i)!} h^{j+n-i} +$$

$$+L \sum_{j=0}^{n-2} \frac{e_k^{(j+1)}}{(j+n-i)!} h^{j+n-i} + L^2 \sum_{j=0}^{n} \frac{e_k^{(j)}}{(j+2n-i-1)!} h^{j+2n-i-1} +$$

$$L^2 \sum_{j=1}^{n} \frac{e_k^{(j)}}{(j+2n-i-2)!} h^{j+2n-i-2} + b_i w_0(h) h^{2n-i}$$

where $i - 0(1)n-1$, and

$$b_i = \frac{L}{(2n-i)!} + \frac{L^2}{(3n-i-1)!} + \frac{L^2}{(3n-i-2)!}.$$

If we use Lemma 2.1, then the inequality (2.1) takes the general form

(2.2)

$$e_{k+1}^{(i)} \le \sum_{j=0}^{n-1} V_{ij} e_k^{(j)} + b_i w_0(h) h^{2n-i},$$

where

$$V_{ij} = \begin{cases} (1 + a_{ij} h) & \text{if } i = j \\ a_{ij} h^{j-i} & \text{if } i < j \\ a_{ij} h^{n-i} & \text{if } j = 0, 1 \text{ and } i > j \\ a_{ij} h^{n-i+j-1} & \text{if } j \ge 2 \text{ and } i > j \end{cases}$$

and a_{ij} are constants independent of h.

By using the Definition 2.2, then the inequality (2.2) for $i = 0(1)n-1$ takes the matrix form.

$$\begin{bmatrix} e_{k+1}^{(0)} \\ e_{k+1}^{(1)} \\ e_{k+1}^{(2)} \\ \vdots \\ e_{k+1}^{(n-1)} \end{bmatrix} \le$$

$$\begin{bmatrix} (1 + a_{00} h) & a_{01} h & a_{02} h^2 & \dots & a_{0\,n-1} h^{n-1} \\ a_{10} h^{n-1} & (1 + a_{11} h) & a_{12} h & \dots & a_{1\,n-1} h^{n-2} \\ a_{20} h^{n-2} & a_{21} h^{n-2} & (1 + a_{22} h) & \dots & a_{2\,n-1} h^{n-3} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ a_{n-1\,0} h & a_{n-1\,1} h^2 & a_{n-1\,2} h^3 & \dots & (1 + a_{n-1\,n-1} h) \end{bmatrix}$$

$$\begin{bmatrix} e_k^{(0)} \\ e_k^{(1)} \\ e_k^{(2)} \\ \vdots \\ e_k^{(n-1)} \end{bmatrix} + b^* w_0(h) h^{n+1} \begin{bmatrix} h^{n-1} \\ h^{n-2} \\ \vdots \\ h \\ 1 \end{bmatrix}$$

Let us denote

$$E_k = (e_k^0, e_k^{(1)}), \dots, e_k^{(n)})^r$$

and briefly write in the form:

(2.3)

$$E_{k+1} \leq (I_n + hA)E_k + w_0(h)h^{n+1}B$$

where I_n is the n-th order unit matrix and $A = [a_{ij}], i, j = 0(1)n-1$ and B is the $(n \times 1)$ matrix, $B = (b^* b^* \dots b^*)^T$. Obviously $b^* = \max\limits_{0 \leq i \leq n-1} b_i$.

Applying the principle of successive substitution, (2.3) reduces to

$$E_{k+1} \leq (I_n + hA)E_0 + w_0(h)h^{n+1}B \sum_{j=0}^{k} (I_n + hA)^j$$

Since $E_0 = \underline{0}$, then we easily get

$$E_{k+1} \leq w_0(h)h^n C$$

where C is and $(n \times 1)$ matrix whose elements are constants independent of h and thus the proof is complete.

Theorem 2.2

Let $\overline{y}_{k+1}^{(n)}$ be the approximate value of $y_{k+1}^{(n)}$ and be given from the formula (1.11). Then the inequality

$$e_{k+1}^{(n)} \leq c_n w_0(h) h^n$$

holds true for all $k = 0(1)N - 1$, where c_n is a constant independent of h.

Proof

Using Lemma 2.1 and the results of theorem 2.1, it is easy to get the proof.

As a conclusion of theorems 2.1 and 2.2, we have proved that the inequality

$$e_{k+1}^{(i)} \leq c_i w_0(h) h^n$$

holds true for all $i = 0(1)n$ and all $k = 0(1)N - 1$.

3. Spline function approximating the solution.

In this paragraph we construct the spline function approximating the solution of (1.1)-(1.2) and we prove the existence and uniqueness of this approximate spline solution. Thus we introduce the following theorem:

Theorem 3.1

Given a mesh of ponts

$$\Delta : 0 = x_0 < x_1 < \ldots < x_k < x_{k+1} < \ldots < x_N = 1$$

where,

$$x_{k+1} - x_k = h, k = 0(1)N - 1$$

and sets of approximate values

$$\overline{Y}^{(t)} : \overline{y}_0^{(t)}, \overline{y}_1^{(t)}, \ldots, \overline{y}_N^{(t)}, t = 0(1)n$$

then, there exists a unique spline function $S_\Delta(x) \equiv S_\Delta(\overline{Y}^{(t)}, x)$ interpolating the set $\overline{Y}^{(t)}$ on Δ and satisfying the following conditions:

(3.1)
$$S(\overline{Y}^{(t)}, x) \equiv S_\Delta(x) \in C^n[0,1],$$

(3.2)
$$S_\Delta^{(t)}(x_k) = \overline{y}_k^{(t)}, \quad S_\Delta^{(t)}(x_N) = \overline{y}_N^{(t)}$$

where $t = 0(1)n, k = 0(1)N - 1$ and for $x_k \le x < x_{k+1}$,
(3.3)
$$S_\Delta(x) \equiv S_k(x) = \sum_{j=0}^{n} \frac{\overline{y}_k^{(j)}}{j!}(x - x_k)^j + \sum_{p=1}^{n+1} a_p^{(k)}(x - x_k)^{p+n}$$

where $a_p^{(k)}$ are constants to be determined.

Proof

From the continuity condition (3.1), using (3.2) and (3.3) it is easy to get
(3.4)
$$\sum_{p=1}^{n+1} t!\binom{p+n}{t} a_p^{(k)} h^{p-1} = F_t^{(k)}$$

where,
(3.5)
$$F_t^{(k)} = h^{t-n-1}\left(\overline{y}_{k+1}^{(t)} - \sum_{j=0}^{n-t} \frac{\overline{y}_k^{(j+t)}}{j!} h^j\right),$$

$t = 0(1)n, k = 0(1)N-1, p = 1(1)n+1$ and $a_p^{(k)}$ are the unknowns to be determined. The system of linear equations (3.4) in the unknowns $a_p^{(k)}$ has a unique solution since its determinant $D \neq 0$. Here,

$$D = | d_{ij} |, \qquad i, j = 1(1)n+1,$$

$$d_{ij} = \binom{n+j}{i-1}(i-1)!h^{j-1}$$

and it is easy to prove that

$$D = h^{n(n+1)/2} \prod_{t=0}^{n} t!$$

and this does not equal to zero since $h \neq 0$.

If we replace the p-th column in D by the column

$$(F_0^{(k)} F_1^{(k)} \dots F_n^{(k)})^T$$

and denote the resulting determinant by D^P, then the solution of the system (3.4) becomes

(3.6)

$$a_p^{(k)} = \frac{D^P}{D}, \quad p = 1(1)n+1$$

and after factorizing D^P in terms of $F_0^{(k)}, F_1^{(k)}, \dots, F_n^{(k)}$, the solution (3.6) takes the form

(3.7)

$$a_p^{(k)} = \frac{1}{h^{p-1}} \sum_{t=0}^{n} c_{pt} F_t^{(k)}$$

where c_{pt} are constants independent of $h, k = 0(1)N-1$, and $p = 1(1)n+1$.

The uniqueness of this solution guaratees the uniqueness of the spline function $S_\Delta(x)$ and xonsequently the existance of this function and this completes the proof.

4. Convergence of the spline function to the solution.

Before we prove theorems dealing with the convergence of the spline function constructed in theorem 3.1 to the exact solution of the differential equation in consideration, we prove the following lemma.

Lemma 4.1

The inequality

$$| a_p^{(k)} | \leq \frac{A_p}{h^p} \omega_0(h)$$

holds true for all $p = 1(1)n + 1$ and $k = 0(1)N - 1$, where A_p are constants independent of h.

Proof

From (3.7), we have

(4.1)

$$| a_p^{(k)} | \leq \frac{1}{h^{p-1}} \sum_{t=0}^{n} c_{pt} \, | F_t^{(k)} |$$

Since the Taylor expansion of $y^{(t)}(x)$ at any point $x = x_{k+1}$ of Δ is given by

(4.2)

$$y_{k+1}^{(t)} = \sum_{j=0}^{n-t-1} \frac{y_k^{(j+t)}}{j!} h^j + \frac{y^{(n)}(\xi_{kt})}{(n-t)!} j^{n-t}$$

where, $t = 0, (1)n, k = 0(1)N - 1$ and $x_k < \xi_{kt} < x_{k+1}$ then by using (4.2) and (3.5), we get

(4.3)

$$| F_t^{(k)} | \leq h^{t-n-1}\{e_{k+1}^{(t)} + \sum_{j=0}^{n-t} \frac{e_k^{(j+t)}}{j!} h^j +$$

$$+\frac{1}{(n-t)!} | y^{(n)}(\xi_{kt}) - y_k^{(n)} | h^{n-t}\}.$$

Using Theorems 2.1 and 2.2 and the definition of modulus of continuity of $y^{(n)}(x)$, then (4.3) becomes

(4.4)

$$| F_t^{(k)} | \leq c_t^* \frac{w_0(h)}{h}, \quad t = 0(1)n$$

where c_t^* are constants independent of h.

Substituting form (4.4) in (4.1), we easily complete the proof of the lemma.

Theorem 4.1

Let $y(x)$ be the exact solution of the differential equation (1.1)-(1.2) and let $f \in C([0,1] \times R^2)$. If $S_\Delta(x)$ is the spline function constructed in theorem 3.1, then there exists a constant λ independent of h, such that

$$| y^{(t)}(x) - S_\Delta^{(t)}(x) | \leq \lambda w_0(h) h^{n-t}$$

is true for all $x \in [0,1]$ and all $t = 0(1)n$.

Proof

Using the Taylor expansion of $y^{(t)}(x)$, (3.3) and the definition of modulus of continuity of $y^{(n)}(x)$, we get

$$| y^{(t)}(x) - S_\Delta^{(t)}(x) | \leq \sum_{j=0}^{n-t} \frac{e_k^{(j+t)}}{j!} h^j + \frac{1}{(n-t)!} w_0(h) h^{n-t} +$$

$$+\sum_{p=1}^{n+1} t!\binom{p+n}{t} \mid a_p^{(k)} \mid h^{p+n-t}.$$

Using Theorems 2.1 and 2.2 and Lemma 4.1, the above inequality becomes

$$\mid y^{(t)}(x) - S_\Delta^{(t)}(x) \mid \leq c_t^{**} w_0(h) h^{n-t}$$

where c_t^{**} are constants independent of h and $t = 0(1)n$.
Let $\lambda = \max_{0 \leq t \leq n} c_t^{**}$, then we get

$$\mid y^{(t)}(x) - S_\Delta^{(t)}(x) \leq \lambda w_0(h) h^{n-t}$$

where λ is a constant independent of h and thus the proof is complete.

Theorem 4.2

If $\overline{S}_\Delta^{(n)}(x)$ denotes the function

$$\overline{S}_\Delta^{(n)}(x) = f(x, S_\Delta(x), S'_\Delta(x))$$

where $S_\Delta(x)$ is the spline function approximating the solution of the differential equation (1.1)-(1.2) and constructed in Theorem 3.1, then the inequality

$$\mid \overline{S}_\Delta^{(n)} - S_\Delta^{(n)}(x) \mid \leq M^* w_0(h)$$

is true for all $x \in [0, 1],$, where M^* is a constant independent of h.
Otherwise,

$$\lim_{h \to 0} S_\Delta^{(n)}(x) = f(x, S_\Delta(x), S'_\Delta(x))$$

Proof

We have

$$| \bar{S}_{\Delta}^{(n)}(x) - S_{\Delta}^{(n)}(x) | \leq | \bar{S}_{\Delta}^{(n)}(x) - y^{(n)}(x) | +$$
$$+ | y^{(n)}(x) - S_{\Delta}^{(n)}(x) |$$

Using the definition of $\bar{S}_{\Delta}^{(n)}$ this becomes

$$= | f(x), S_{\Delta}(x), S'_{\Delta}(x)) - f(x, y(x), y'(x)) | +$$
$$+ | y^{(n)}(x) - S_{\Delta}^{(n)}(x) |$$

By using the Lipschitz condition (1.4) and Theorem 4.1, we get

$$| \bar{S}_{\Delta}^{(n)}(x) - S_{\Delta}^{(n)}(x) | \leq (L\lambda h^{n} + L\lambda h^{n-1} + \lambda) w_0(h)$$
$$\leq M^{*} w_0(h)$$

where M^{*} is a constant independent of h and thus the proof is complete.

5. Stability of the method for the n-th order differential equation.

In this paragraph, we are going to prove the stability of the method for approximating the solution of the differential equation in consideration and described in the previous sections. We begin with the following definition:

Definition 5.1

If any of the calculated values $\bar{y}_k^{(i)}$ is changed to $\bar{z}_k^{(i)}$, where $i = 0(1)n$, then the introduced error is defined by

$$\epsilon_k^{(i)} = | \bar{z}_k^{(i)} - \bar{y}_k^{(i)} |, i = 0(1)n.$$

This change will impose a change, at any point $x = x_{m+1}$, in the calculated values from $\bar{y}^{(i)}_{m+1}$ to $\bar{z}^{(i)}_{m+1}$, where $m = k(1)N - 1$. This leads us to solve

(5.1)

$$\bar{z}^{(i)}_{m+1} = \sum_{j=0}^{n-i-1} \frac{\bar{z}^{(i+j)}_m}{j!} h^j + \underbrace{\int_{x_m}^{x_{m+1}} \cdots \int_{x_m}^{t_{n-i-1}}}_{(n-i) \text{ times}} f(t_{n-i}, z^*_m(t_{n-i}), z^{**'}_m(t_{n-i}))dt_{n-i} \ldots dt_1$$

instead of (1.6) where $i = 0(1)n - 1$,

(5.2)

$$z^*_m(t) = \sum_{j=0}^{n} \frac{\bar{z}^{(j)}_m}{j!}(t - x_m)^j, x_m \leq t \leq x_{m+1},$$

(5.3)

$$z^{**'}_m(t) = \sum_{j=0}^{j-2} \frac{\bar{z}^{(j+1)}_m}{j!}(t - x_m)^j + \underbrace{\int_{x_m}^{t} \cdots \int_{x_m}^{t_{n-2}}}_{(n-1) \text{ times}} f(t_{n-1}, z^*_m(t_{n-1}), z^{*'}_m(t_{n-1})).dt_{n-1} \ldots dt_1$$

and

(5.4)

$$z^{*'}_m(t) = \sum_{j=0}^{n-1} \frac{\bar{z}^{(j+1)}_m}{j!}(t - x_m)^j, x_m \leq t \leq x_{m+1}.$$

Also $\bar{z}^{(n)}_{m+1}$ can be defined as
(5.5)

$$\bar{z}^{(n)}_{m+1} = f(x_{m+1}, \bar{z}_{m+1}, \bar{z}'_{m+1}).$$

In the following theorems we prove that the error is a multiple bound of the introduced error.

Theorem 5.1

If any of the calculated values $\bar{y}^{(i)}_k$ is changed to $\bar{z}^{(i)}_k, i = 0(1)n-1$, then the inequality

$$\epsilon^{(i)}_{m+1} = |\, \bar{z}^{(i)}_{m+1} - \bar{y}^{(i)}_{m+1} \,| \leq u_i \epsilon^*_k$$

holds for all $m = k, k+1, \ldots, N-1$ and $i = 0, 1, \ldots, N-1$, where

$$\epsilon^*_k = \max\{\epsilon^{(0)}_k, \epsilon^{(1)}_k, \ldots, \epsilon^{(n-1)}_k\},$$

$\epsilon^{(i)}_k$ are the introduced error and u_i are constant independent of h.

Proof

By using (5.1),(1.6) and the Lipschitz condition (1.4), we get
(5.6)

$$\epsilon^{(i)}_{m+1} \leq \sum_{j=0}^{n-i-1} \frac{\epsilon^{(i+j)}_m}{j!} h^j + L \sum_{j=0}^{n} \frac{\epsilon^{(j)}_m}{(j+n-i)!} h^{j+n-i} +$$

$$+ L \sum_{j=0}^{n-2} \frac{\epsilon^{(j+1)}_m}{(j+n-i)!} h^{j+n-i} + L^2 \sum_{j=0}^{n} \frac{\epsilon^{(j)}_m}{(j+2n-i-1)!} h^{j+2n-i-1} +$$

$$L^2 \sum_{j=1}^{n} \frac{\epsilon^{(j)}_m}{(j+2n-i-2)!} h^{j+2n-i-2}.$$

Since $\epsilon_m^{(n)} = | \bar{z}_m^{(n)} - \bar{y}_m^{(n)} |$
then by using (5.5),(1.11) and the Lipschitz condition (1.4), we get
(5.7)

$$\epsilon_m^{(n)} \leq L\epsilon_m + L\epsilon'_m$$

By using (5.7), then (5.6) can be written in the general form as:
(5.8)

$$\epsilon_{m+1}^{(i)} \leq \sum_{j=0}^{n-1} w_{ij}\epsilon_m^{(j)}$$

where

$$w_{ij} = \begin{cases} (1 + q_{ij}h) & \text{if } i = j \\ q_{ij}h^{j-i} & \text{if } i < j \\ q_{ij}h^{n-i} & \text{if } j = 0,1 \text{ and } i > j \\ q_{ij}h^{n-i+j-1} & \text{if } j \geq 2 \text{ and } i > j \end{cases}$$

and q_{ij} are constants independent of h.

If we use the Definition 2.2, then the inequality (5.8) for $i = 0(1)n-1$ takes the following matrix form.
(5.9)

$$\Psi_{m+1} \leq (I_n + hQ)\Psi_m$$

where I_n is the n-th order unit matrix, $Q = [q_{ij}], i, j = 0(1)n-1$,

$$\Psi_{m+1} = (\epsilon_{m+1}^{(0)} \epsilon_{m+1}^{(1)} \dots \epsilon_{m+1}^{(n-1)})^T$$

and

$$\Psi_m = (\epsilon_m^{(0)} \epsilon_m^{(1)} \dots \epsilon_{m+1}^{(n-1)})^T.$$

The principle of the successive substitution implies

$$\begin{aligned}
\Psi_{m+1} \leq & \quad (I_n + hQ)\Psi_m \\
\Psi_m (I_n + hQ) \leq & \quad (I_n + hQ)^2 \Psi_{m-1}, \\
\Psi_{m-1}(I_n + hQ)^2 \leq & \quad (I_n + hQ)^3 \Psi_{m-2}, \\
\cdots & \quad \cdots \\
\Psi_{k+1}(I_n + hQ)^{m-k} \leq & \quad (I_n + hQ)^{m-k+1}\Psi_k.
\end{aligned}$$

Hence

$$\begin{aligned}
\Psi_{m+1} \leq & \quad (I_n + hQ)^{m-k+1}\Psi_k \\
\leq & \quad (I_n + \frac{1}{n}Q)^N \Psi_k \\
\leq & \quad e^Q \Psi_k = U\Psi_k
\end{aligned}$$

where

$$\Psi_k = (\epsilon_k^{(0)} \epsilon_k^{(1)} \ldots \epsilon_k^{(n-1)})^T$$

and U is $(nx1)$ matrix whose elements are constants independent of h.
i.e.:

$$\epsilon_{m+1}^{(i)} \leq u_i \epsilon_k^*$$

where $i = 0(1)n-1$,

$$\epsilon_k^* = \max\{\epsilon_k^{(0)}, \epsilon_k^{(1)}, \ldots, \epsilon_k^{(n-1)}\}.$$

Hence the proposition of the theorems.

<u>Theorem 5.2</u>

The inequality

$$\epsilon_{m+1}^{(n)} = \mid \bar{z}_{m+1}^{(n)} - \bar{y}_{m+1}^{(n)} \mid \leq u_n \epsilon_k^*$$

is true for all $m = k, k+1, \ldots, N-1$, where u_n is a constant independent of h and

$$\epsilon_k^* = \max\{\epsilon_k^{(0)}, \epsilon_k^{(1)}, \ldots, \epsilon_k^{(n-1)}\}.$$

Proof

Using the inequality (5.7) and the results of Theorem 5.1, it is easy to get the proof.

Theorems 5.1 and 5.2 imply

(5.10)
$$| \bar{z}_m^{(t)} - \bar{y}_m^{(t)} | \leq u_t \epsilon_k^*$$

which holds for all $m = k, k+1, \ldots, N-1$ and all $t = 0, 1, \ldots, n$.

Theorem 5.3

If any of the calculated values $\bar{y}_k^{(i)}$ is changed to $\bar{z}_k^{(i)}$, where $i = 0(1)n-1$, and consequently, the spline function approximating the solution of the differential equation (1.1)-(1.2), and constructed in theorem 3.1 is also changed from $S_\Delta(x)$ to $s_\Delta(x)$, then for any $x \in [x_m . x_{m+1}], m = k, k+1, \ldots, N-1$, the inequality

$$| s_\Delta^{(t)}(x) - S_\Delta^{(t)}(x) | \leq u^{**} \epsilon_k^*$$

holds ture for all $t = 0, 1, \ldots, n$, where u^{**} is a constant independent of h and $\epsilon_k^* = \max\{\epsilon_k^{(0)}, \epsilon_k^{(1)}, \ldots, \epsilon_k^{(n-1)}\}$.

Proof

The new spline function, due to the variation from $\bar{y}_k^{(i)}$ to $\bar{z}_k^{(i)}$ and using (3.3), will be

(5.11)

$$s_m(x) = \sum_{j=0}^{n} \frac{\bar{z}_m^{(j)}}{j!}(x - x_m)^j + \sum_{p=1}^{n+1} b_p^{(m)} (x - x_m)^{p+n}$$

and satisfying the following conditions, as of (3.1) and
(5.12)

$$s_{\Delta}(x) \in c^{n}[0,1],$$

(5.13)

$$s_{m}^{(t)}(x_{m}) = \bar{y}_{m}^{(t)}, s_{N-1}^{(t)}(x_{N}) = \bar{y}_{N}^{(t)}.$$

For (5.11), (5.12) and (5.13), we get
(5.14)

$$\bar{z}_{m+1}^{(t)} = \sum_{j=0}^{n-t} \frac{\bar{z}_{m}^{(j+t)}}{j!} h^{j} + \sum_{p=1}^{n+1} t! \binom{p+n}{t} b_{p}^{(m)} h^{p+n-t}.$$

The system of equations corresponding to (3.4) becomes
(5.15)

$$\sum_{p=1}^{n+1} t! \binom{p+n}{t} b_{p}^{(m)} h^{p-1} = G_{t}^{(m)}$$

where $t = 0(1)n$ and
(5.16)

$$G_{t}^{(m)} = h^{t-n-1} \{\bar{z}_{m+1}^{(t)} - \sum_{j=0}^{n-t} \frac{\bar{z}_{m}^{(j+t)}}{j!} h^{j}\}$$

and analouge to (3.7), we get
(5.17)

$$b_{p}^{(m)} = \frac{1}{h^{p-1}} \sum_{t=0}^{n} c_{pt} G_{t}^{(m)}.$$

From (5.11) and (3.3), we get
(5.18)

$$| s_{m}^{(t)}(x) - S_{m}^{(t)}(x) \leq$$

$$\sum_{j=0}^{n-t} \frac{| \bar{z}_m^{(j+t)} - \bar{y}_m^{(j+t)} |}{j!} h^j + \sum_{p=1}^{n+1} t! \binom{p+n}{t} | b_p^{(m)} - a_p^{(m)} | h^{p+n-t}$$

From(5.17) and (3.7), we have
(5.19)

$$| b_p^{(m)} - a_p^{(m)} | \leq \frac{1}{h^{p-1}} \sum_{t=0}^{n} c_{pt} | G_t^{(m)} - F_t^{(m)} |$$

and from (5.16) and (3.5), we have

$$| G_t^{(m)} - F_t^{(m)} | \leq h^{t-n-1} \{ | \bar{z}_{m+1}^{(t)} - \bar{y}_{m+1}^{(t)} | +$$

$$+ \sum_{j=0}^{n-t} \frac{| \bar{z}_m^{(j+t)} - \bar{y}_m^{(j+t)} |}{j!} h^j \}.$$

Applying Theorem 5.1 and 5.2, we get

$$| G_t^{(m)} - F_t^{(m)} | \leq u^* \epsilon_k^* h^{t-n-1}$$

where u^* is a constant independent of h.
And so, (5.19) becomes

$$| b_p^{(m)} - a_p^{(m)} | \leq \frac{1}{h^{p-1}} \sum_{t=0}^{n} c_{pt} u^* \epsilon_k^* h^{t-n-1}.$$

Thus, from Theorems 5.1, 5.2 and the above inequality, we get

$$| s_m^{(t)}(x) - S_m^{(t)}(x) | \leq [\sum_{j=0}^{n-t} u_t \frac{h^j}{j!} +$$

$$+ \sum_{p=1}^{n+1} \sum_{t=0}^{n} t! \binom{p+n}{t} c_{pt} u^*] \epsilon_k^* \leq u^{**} \epsilon_k^*$$

where u^{**} is a constant independent of h.
Hence the proposition.

Numerical Examples.

In this paragraph we test our method, numerically, by the following two examples:

Example 1.

$$y'' = y' + 1, x_0 = 0, y_0 = 1, y_0' = 1.$$

The exact solution is:

$$y(x) = 2e^{-x} - x$$

The following table presents the results

	Exact value	Numerical value	Absolute Error
$y(0.4)$	1.5.836494	1.583642249	$7.151E-6$
$y'(0.4)$	1.9836494	1.983601391	$4.8009E-5$
$y''(0.4)$	2.9836494	2.983601391	$4.8009E-5$

Example 2.

$$y''' = -y - x.x_0 = 0, y_0 = 1, y_0' = -2, y_0'' = 1.$$

The exact solution is:

$$y(x) = e^{-x} - x$$

The following table presents the results

	Exact value	Numerical value	Absolute Error
$y(0.4)$	0.270320044	0.270320045	$1.0E-9$
$y'(0.4)$	−1.670320044	−1.670319999	$4.5E-8$
$y''(0.4)$	0.670320044	0.670320328	$2.84E-7$
$y'''(0.4)$	−0.670320044	−0.670320045	$1.0E-9$

References

[1] **Roman, V.M.** A numerical solution of n-th order differential equations using cubic splines. Ukrain. Mat. v.32(1980), no.5, 686-693.

[2] **Micula, G.H.** Spline functions approximating the solution of non linear differential equation of n-th order. Y. Angew, Math. Mech. 52(1972), 189-190.

[3] **Arndt, Herbert** Lösung von gewöhnlichen differential gleichungen mit nichtlinearen splines. Numer. Math. 33(1979), no.3, 323-338.

[4] **Patricio, F.** A numerical method for solving initial-value problems with spline functions. BIT 19(1979), no.4, 489-494.

[5] **Brianzi, P.; Rebolia, L.** Numerical characteristica of the integral form of differential equations of order n. Calcolo 19(1982), no.1, 71-86.

[6] **Tharwat Fawzy** Spline Functions and Cauchy Problems, VI. Annales Univ. Sci. Budapest, Sectio Math. Tom.27 (1986), 3-10.

Received: July 10. 1987.