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AN ACCURATE DOUBLE CHEBYSHEV SPECTRAL APPROXIMATION FOR POISSON'S EQUATION

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Abstract. A double Chebyshev spectral method for the solution of Poisson's equation in a square subject to the most general mixed boundary conditions, based on the relation between the Chebyshev coefficients for the solution and those for the right-hand side, is developed. The linear system of equations for the expansion coefficients is greatly simplified by using the Kronecker matrix algebra. The accuracy and efficiency of this Chebyshev approach compare favorably with those of the standard finite-difference methods. The extension of the method to Helmholtz equation is also described. Efficient evaluation schemes for function values and derivatives are also presented.

1. Introduction

Many physical problems require a numerical solution of the two-dimensional Poisson's equation. On rectangular domains, the fast Poisson solvers provide a rapid solution of the standard five-point difference approximation to the partial differential equation

(See, for instance, Vichnevtsky (1981), and the references given there). The resolution of these methods is limited by their algebraic convergence, i.e., provided that the solution of the continuous problem has continuous and bounded fourth order partial derivatives, the maximum error of the discrete approximation with $N+1$ grid points in each direction decays as $1/N^2$.

A fourier method for solving numerically Poisson's equation in a rectangle subject to inhomogeneous Dirichlet boundary conditions is discussed by Skõllermo (1975); the method based on the relation between the fourier coefficients for the solution and those for the right-hand side of the equation being solved. The fast fourier transform is used for the computation (cf. The reviews of Dorr (1970) and Swarztrauber (1977)). The method is shown to be second order accurate under certain conditions on the smoothness of the solution. The accuracy of this method is found to be limited by the lack of smoothness of the periodic extension of the inhomogeneous term.

Spectral methods involve seeking the solution to a differential equation in terms of a series of known smooth functions. They have emerged as a viable alternative to finite difference and finite element methods for the numerical solution of partial differential equations. The essence of the spectral approach is the expansion of the solution into a truncated series. The convergence of this approximation is governed by the rate of decay of the expansion coefficients. The order of magnitude of these coefficients can be estimated by repeated application of integration by parts. If the basis functions are suitably chosen, the boundary terms from each integration by parts will vanish and thus one power of $1/n$ will be added to the estimate of the n-th coeffcient. If the solution is infinitely differentiable, then the expansion coefficients will decrease faster than any finite power of $1/n$. Thus, the error made by retaining only a finite number N of the terms of the series will itself decrease faster than any finite power of $1/N$, which means that the convergence of the spectral approximation will have an exponential rather than an algebraic character. On the other hand, if the solution has only a finite number of derivatives, then the spectral approximation is expected to converge algebraically like some finite power of $1/N$.

Chebyshev polynomials - as a basis in the spectral expansion - have been used in the solution of differential equations by many authors, including, for ordinary differential equations, Lanczos (1957), Clenshaw (1957, 1962), Clenshaw & Elliot (1960), Fox & Parker (1972), Snell (1970), Boateng (1975), Morris & Horner (1977), Horner (1980), and for partial differential equations, Elliot (1961), Scraton (1965), Mason (1967a, 1967b, 1969, 1984), Knibb & Scraton (1971), Dew & Scraton (1973, 1975), Knibb (1975), Doha (1979, 1983, 1984, 1989), Maday & Quarteroni (1981) and Tal-Ezer (1989).

Poisson's equation in a square with homogeneous and inhomogeneous Dirichlet boundary conditions have been considered by Gottlieb & Országh (1977), Haidvogel & Zang (1979) and Horner (1982).

In the present paper we develop a spectral method based on an expansion in doubly Chebyshev polynomials for solving Poisson's equation in two-space variables in a square subject to the most general inhomogeneous mixed boundary conditions. Some important results about functions of one and two variables in terms of Chebyshev polynomials are given in Sec.2. The formulation and derivation of the method of solution is described in Sec. An alternative method of solution is explained in Sec.4; the extension of the methods to Helmholtz equation is noted at the end of this section. Several numerical results and comparisons are discussed in Sec.5. Some concluding remarks are given in Sec.6.

The motivation for adopting this approach is that it provides one with a semi-analytical solution which not only compares favorably to other analytic type methods but also has some advantages over the solution obtained by using the standard finite difference and finite element methods. Some of these advantages are :

(i) Once the expansion series is known, the values of the solution and its derivatives can be found globally and not restricted to points of a grid as in the finite difference methods.

(ii) The mathematical features of the proposed methods follow very closely those of the differential equation being solved. Thus, the boundary conditions imposed on the methods of solution are normally the same as those imposed on the differential equation.In contrast, finite difference methods of higher order than the differential equation require additional boundary conditions. Many of the complications of finite order finite difference methods disappear with the infinite order accurate semi- analytical methods.

(iii) If the solution is infinitely differentiable, then the proposed method has the property that its error goes to zero faster than any finite power of the number of retained modes. In contrast, finite difference and finite element methods yield finite order rates of convergence. The important consequence is that the proposed methods can achieve high accuracy with little more resolution than is required to achieve moderate accuracy.

2. Some important results about functions of one and two variables in terms of Chebyshev polynomials

Let the function $f(x)$ and its derivatives $f^{(1)}, f^{(2)}, \ldots$, be expressed as uniformly covergent series in the form

$$f^{(k)}(x) = {\sum_{i=0}^{\infty}}' a_i^{(k)} T_i(x) \qquad k = 0, 1, 2, \ldots \tag{1}$$

where $a_i^{(k)}$ are constants, with the superscript to indicate the order of the derivative $f^{(k)}(x)$; here $\sum'$ denotes halving the first term in the series. Then the coefficients in the series for successive derivatives can be related by

$$a_{i-1}^{(k)} - a_{i+1}^{(k)} = 2ia_i^{(k-1)} \; ; \tag{2}$$

when ordinary differential equations are solved in Chebyshev series, the general equation (2) can be used with another equation derived from the particular equation being solved, in order to find the coefficients in the series solution, (see Clenshaw (1957), Horner (1980) and Wimp (1984)). In Sec.3, it is seen how these ideas can be extended to certain partial differential equations.

Following Clenshaw (1962), Smith (1965), Hunter (1970) and Horner (1980) the following formulae can be found to evaluate the finite sum

$$\sum_{i=0}^{n}{}' a_i^{(k)} T_i(x) \qquad \text{representing an approximation to } f^{(k)}(x).$$

$$\left.\begin{array}{lll} \text{With} & b_{r+1}^{(-1)} = \frac{1}{2}a_r, & r = 0,1,2,\ldots,n \\ \text{let} & b_{n+2-k}^{(k)} = b_{n+1-k}^{(k)} = 0 & \\ \text{and calculate} & & \\ & b_r^{(k)} = 2b_{r-1}^{(k-1)} + 2xb_{r+1}^{(k)} - b_{r+2}^{(k)}, & r = n-k,\ldots,1,0 \\ \text{Then} & \frac{f^{(k)}(x)}{k!} = \frac{1}{2}(b_0^{(k)} - b_2^{(k)}), & k = 0,1,\ldots,n \end{array}\right] \tag{3}$$

Analogous results are now given for functions of two variables.

Let the function $f(x,y)$ be expressed as a uniformly convergent double series of Chebyshev polynomials in the form

$$f(x,y) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty}{}'' a_{ij} T_i(x) T_j(y) \; . \tag{4}$$

Basu (1973) refers to series (4) as a bivariate Chebyshev series expansion. The double primes in (4) indicate that the first term is $\frac{1}{4}a_{00}$; a_{m0} and a_{0n} are to be taken as $\frac{1}{2}a_{m0}$ for $m > 0, n > 0$ respectively.

Now if we write

$$b_i = \sum_{j=0}^{\infty}{}' a_{ij} T_j(y) \ , \quad c_j = \sum_{i=0}^{\infty}{}' a_{ij} T_i(x) \ ,$$

then (4) can be interpreted

$$f(x,y) = \sum_{i=0}^{\infty}{}' b_i T_i(x) \ , \qquad (5)$$

$$\text{or} \quad f(x,y) = \sum_{j=0}^{\infty}{}' c_j T_j(y) \ . \qquad (6)$$

Basu gives the two-dimensional analouge of (3), with $k = 0$. Using his notation, results for function and derivative values are given below. Thus, as in (5), let

$$S = S_{mn}(x,y) = \sum_{i=0}^{m} \sum_{j=0}^{n}{}'' a_{ij} T_i(x) T_j(y) = \sum_{i=0}^{m}{}' b_i T_i(x) \ ,$$

$$\text{where} \quad b_i = \sum_{j=0}^{n}{}' a_i j T_j(y) \ .$$

Let

$$\left.\begin{aligned} g_{m+2} &= g_{m+1} = 0, \text{ and } d_{i,n+2} = d_{i,n+1} = 0, \\ d_i j &= a_{ij} + 2y d_{i,j+1} - d_{i,j+2}, \; j = n, \ldots, 0 \\ b_i &= \frac{1}{2}(d_{i,0} - d_{i,2}) \\ g_i &= b_i + 2x g_{i+1} - g_{i+2} \\ \text{then } S &= S_{mn}(x,y) = \frac{1}{2}(g_0 - g_2) \end{aligned}\right] \quad i = m, \ldots, 0. \quad (7)$$

Alternatively, as in (6), let $S = S_{mn}(x,y) = \sum_{j=0}^{n}{}' c_j T_j(y)$,

$$\text{where} \quad c_j = \sum_{i=0}^{m}{}' a_{ij} T_i(x).$$

Let

$$\left.\begin{aligned} g_{n+2} &= g_{n+1} = 0, \text{ and } d_{m+2,j} = d_{m+1,j} = 0, \\ d_{i,j} &= a_{ij} + 2x d_{i+1,j} - d_{i+2,j}, \quad i = m, \ldots, 0 \\ c_j &= \frac{1}{2}(d_{0,j} - d_{2,j}), \\ g_j &= c_j + 2y g_{j+1} - g_{j+2}, \\ \text{then } S &= S_{mn}(x,y) = \frac{1}{2}(g_0 - g_2). \end{aligned}\right] \quad j = n, \ldots, 0. \quad (8)$$

In calculating S, either of (7) or (8) is satisfactory. In the formulae below, the results are extended in order to evaluate derivatives, with the work based on (7). If we write

$$g_i^{(p,q)} = \frac{1}{p!q!}\frac{\partial^{p+q} g_i}{\partial x^p \partial y^q}, \qquad b_i^{(p,q)} = \frac{1}{p!q!}\frac{\partial^{p+q} b_i}{\partial x^p \partial y^q},$$

$$d_{i,j}^{(p,q)} = \frac{1}{p!q!}\frac{\partial^{p+q} d_{i,j}}{\partial x^p \partial y^q},$$

then differentiation of (7) leads formally to the set of equations

$$\left.\begin{aligned} g_{m+2}^{(p,q)} &= g_{m+1}^{(p,q)} = 0, \quad d_{i,n+2}^{(p,q)} = d_{i,n+1}^{(p,q)} = 0, \\ d_{i,j}^{(p,q)} &= 2d_{i,j+1}^{(p,q-1)} + 2yd_{i,j+1}^{(p,q)} - d_{i,j+2}^{(p,q)}, \\ &\qquad j = n,\ldots,0 \\ b_i^{(p,q)} &= \frac{1}{2}(d_{i,0}^{(p,q)} - d_{i,2}^{(p,q)}), \\ g_i^{(p,q)} &= b_i^{(p,q)} + 2g_{i+1}^{(p-1,q)} + 2xg_{i+1}^{(p,q)} - g_{i+2}^{(p,q)}, \\ S^{(p,q)} &= \frac{1}{2}(g_0^{(p,q)} - g_2^{(p,q)}), \end{aligned}\right] \quad i = m,\ldots,0 \qquad (9)$$

where

$$S^{(p,q)} = \frac{1}{p!q!}\frac{\partial^{p+q} f(x,y)}{\partial x^p \partial y^q}.$$

Moreover, $(d_{i,j})$ and (b_i) are independent of x, and hence for $p > 0$, the above become

$$\begin{aligned} g_{m+2}^{(p,q)} &= g_{m+1}^{(p,q)} = 0 \\ g_i^{(p,q)} &= 2g_{i+1}^{(p-1,q)} + 2xg_{i+1}^{(p,q)} - g_{i+2}^{(p,q)}, \quad i = m,\ldots,0 \qquad (10) \\ S^{(p,q)} &= \frac{1}{2}(g_0^{(p,q)} - g_2^{(p,q)}) \end{aligned}$$

leading to all derivatives, including mixed derivatives, involving differentiation with respect to x. Obvious similarities exist between equations (3) and (10).

In these formulae, the various derivatives are found by letting $p, q = 0, 1, \ldots,$ with the convention that

$$g_i^{(0,0)} = g_i \ ; \ d_i^{(0,0)} = d_i \ ; \ b_i^{(0,0)} = b_i \ ; \ d_{i,j+1}^{(0,-1)} = \frac{1}{2}a_{ij} \ ;$$

$$d_{i,j+1}^{(p,-1)} = g_{i+1}^{(-1,q)} = 0 \quad (p > 0).$$

With $p \neq 0$, equation (10) leads to all derivatives expect those purely with respect to y. These latter derivatives with $p = 0$, $q > 0$, can be obtained through the general scheme (9) above. With $p, q = 0, 1, 2$, the results (7), (9), (10) will be used to find function and derivative values, following the solution of second order partial differential equations.

3. Derivation of the method of solution

In this section we develop a method based on an expansion in Chebyshev polynomials for solving Poisson's equation in two space variables, namely

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = f(x,y) \qquad -1 \leq x, y \leq 1 \tag{11}$$

subject to the most general inhomogeneous mixed boundary conditions

$$\left.\begin{aligned} \alpha_1 u + \beta_1 \frac{\partial u}{\partial x} &= \gamma_1(y) \qquad x = -1 \\ \alpha_2 u + \beta_2 \frac{\partial u}{\partial x} &= \gamma_2(y) \qquad x = 1 \end{aligned}\right] \quad -1 \leq y \leq 1 \tag{12}$$

$$\left.\begin{aligned} \alpha_3 u + \beta_3 \frac{\partial u}{\partial y} &= \gamma_3(x) \qquad y = -1 \\ \alpha_4 u + \beta_4 \frac{\partial u}{\partial y} &= \gamma_4(x) \qquad y = 1 \end{aligned}\right] \quad -1 \leq x \leq 1 \tag{13}$$

It is assumed that the solution to the above problem can be expressed in a uniformly convergent double Chebyshev series expansion

$$u(x,y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty}{}'' a_{mn} T_m(x) T_n(y) \tag{14}$$

where the polynomials $T_m(x)$ can be expressed as

$$T_m(x) = \cos(m \cos^{-1} x) \ .$$

Throughout this paper we assume that there is no discontinuity between the boundary conditions at the four corners of the domain of solution. We also assume that $f(x,y)$ has known bivariate Chebyshev series expansion

$$f(x,y) = \sum_{m=0}^{\infty} {\sum_{n=0}^{\infty}}'' f_{mn} T_m(x) T_n(y)$$

which is uniformly convergent in $-1 \leq x,y \leq 1$. It then follows that the solution (11) has a double series expansion of the form (14), and the solution is free of discontinuities. (The case in which discontinuities are present at the vertices $(\pm 1, \pm 1)$, can often be treated by a method similar to that described by Knibb & Scraton (1971) in the solution of parabolic partial differential equations in Chebyshev series).

Now, let us assume the following expansions

$$\left.\begin{aligned} \frac{\partial^2 u}{\partial x^2} &= \sum_{m=0}^{\infty} {\sum_{n=0}^{\infty}}'' a_{mn}^{(2,0)} T_m(x) T_n(y) \\ \frac{\partial^2 u}{\partial y^2} &= \sum_{m=0}^{\infty} {\sum_{n=0}^{\infty}}'' a_{mn}^{(0,2)} T_m(x) T_n(y) \end{aligned}\right] \qquad (15)$$

where $a_{mn}^{(p,q)}$ denote the Chebyshev expansion coefficients of $(\partial^{p+q} u)/(\partial x^p \partial y^q)$. It is noted here that the sets of coefficients $\{a_{mn}^{(2,0)}\}, \{a_{mn}^{(0,2)}\}$, are constants and same significance in their respective series the sets of coefficients $\{a_i^{(k)}\}$ in the series (1). Now, if we satisfy the differential equation (11), we get

$$a_{mn}^{(2,0)} + a_{mn}^{(0,2)} = f_{mn} \qquad m,n \geq 0 \qquad (16)$$

and applying equation (2) to the series coefficients for the partial derivatives with respect to x, twice gives

$$a_{mn} = \left(\frac{f_{m-2,n} - a^{(0,2)}_{m-2,n}}{4m(m-1)}\right) - \left(\frac{f_{mn} - a^{(0,2)}_{mn}}{2(m-1)(m+1)}\right) + \left(\frac{f_{m+2,n} - a^{(0,2)}_{m+2,n}}{4m(m+1)}\right) \qquad m \geq 2\ , \quad n \geq 0$$

which may be written in the matrix form

$$a_{mn} + \sum_{i=0}^{\infty}{}' A_{mi} a^{(0,2)}_{in} = \sum_{i=0}^{\infty}{}' A_{mi} f_{in} \qquad m \geq 2,\ n \geq 0 \qquad (17)$$

where

$$A_{mi} = \begin{cases} \frac{1}{4}m(m-1) & i = m-2 \\ & \text{(to be doubled if } m = 2) \\ -\frac{1}{2}(m-1)(m+1) & i = m \\ \frac{1}{4}m(m+1) & i = m+2 \\ 0 & \text{otherwise.} \end{cases} \qquad (18)$$

Further use of equation (2), now with regard to the derivative in y, yields the final result

$$\sum_{i=0}^{\infty}{}' A_{mi} a_{in} + \sum_{j=0}^{\infty}{}' a_{mj} B_{jn} = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty}{}'' A_{mi} f_{ij} B_{jn} \qquad m, n \geq 2 \quad (19)$$

where, $B_{jn} = A_{nj}$.

Note that the unknows in the present method are the set of coefficients $\{a_{ij}\}$ in a series expansion, while in the finite difference method are function values $\{u(x,y)\}$ at the points $\{(x,y)\}$ on a grid in the domain of solution. When the coefficients in the present series are known, the global solution can be used to obtain function values, derivatives, (using (7) at any point of the solution domain). In many cases the Chebyshev coefficients $\{a_{ij}\}$ decrease rapidly in magnitude, and a smaller number of them is wanted in a Chebyshev polynomial approximation to the solution $u(x,y)$, than the number of grid points required for similar accuracy using finite difference methods. If suitably large values of m, n are chosen for the polynomial

$$\sum_{m=0}^{M} \sum_{n=0}^{N}{}'' a_{mn} T_m(x) T_n(y)$$

to give a satisfactory approximate solution to equation (11), then the problem becomes an algebraic one of finding the coefficients of the set $\{a_{ij}\}$.

Thus it is assumed that $a_{mn} = 0$, $m > M$, $n > N$. Then letting m range from 2 to M and n from 2 to N , a total of $(M-1)(N-1)$ equations is obtained from (19). The remaining $(M+1)(N+1) - (M-1)(N-1) = 2M + 2N$, equations required for finding the coefficients $\{a_{mn}\}, m = 0, 1, ..., M, n = 0, 1, ..., N$ are obtained from the boundary conditions.

3.1 Use of boundary conditions

If we assume that $\gamma_i(y)$ $(i = 1,2)$ and $\gamma_i(x)$ $(i = 3,4)$ have known Chebyshev expansions

$$\gamma_i(y) = \sum_{n=0}^{\infty}{}' \gamma_n^{(i)} T_n(y) \qquad (i = 1, 2)$$

$$\gamma_i(x) = \sum_{n=0}^{\infty}{}' \gamma_n^{(i)} T_n(x) \qquad (i = 3, 4)$$

then the boundary conditions (12) and (13) give

$$\left.\begin{aligned} \sum_{m=0}^{\infty}{}' (-1)^m (\alpha_1 - m^2\beta_1) a_{mn} &= \gamma_n^{(1)} \\ \sum_{m=0}^{\infty}{}' (\alpha_2 + m^2\beta_2) a_{mn} &= \gamma_n^{(2)} \end{aligned}\right] \quad n = 0, 1, 2, \ldots \tag{20}$$

$$\left.\begin{aligned} \sum_{n=0}^{\infty}{}' (-1)^n (\alpha_3 - n^2\beta_3) a_{mn} &= \gamma_m^{(3)} \\ \sum_{n=0}^{\infty}{}' (\alpha_4 + n^2\beta_4) a_{mn} &= \gamma_m^{(4)} \end{aligned}\right] \quad m = 0, 1, 2, \ldots \tag{21}$$

Equations (20) and (21), after some manipulation, may be written in the finite forms

$$\left.\begin{aligned} \frac{1}{2} a_{0n} + \sum_{m=2}^{M} \mu_m a_{mn} &= g_n \\ a_{1n} + \sum_{m=2}^{M} \nu_m a_{mn} &= h_n \end{aligned}\right] \quad n = 0, 1, 2, \ldots, N \tag{22}$$

$$\left.\begin{aligned} \frac{1}{2} a_{m0} + \sum_{n=2}^{N} U_n a_{mn} &= r_m \\ a_{m1} + \sum_{n=0}^{N} V_n a_{mn} &= s_m \end{aligned}\right] \quad m = 0, 1, 2, \ldots, M \tag{23}$$

where

$$\mu_m = \{(\alpha_1 + m^2\beta_1)(\alpha_2 - \beta_2) + (-1)^m (\alpha_2 + m^2\beta_2)(\alpha_1 - \beta_1)\}/\delta_1$$

$$\nu_m = \{\alpha_2(\alpha_1 + m^2\beta_1) - (-1)^m \alpha_1(\alpha_2 - m^2\beta_2)\}/\delta_1$$

$$g_n = \{(\alpha_2 - \beta_2)\gamma_n^{(1)} + (\alpha_1 + \beta_1)\gamma_n^{(2)}\}/\delta_1$$

$$h_n = \{\alpha_2\gamma_n^{(1)} - \alpha_1\gamma_n^{(2)}\}/\delta_1$$

$$U_n = \{(\alpha_3 + n^2\beta_3)(\alpha_4 - \beta_4) + (-1)^n (\alpha_4 + n^2\beta_4)(\alpha_3 - \beta_3)\}/\delta_2$$

$$V_n = \{\alpha_4(\alpha_3 + n^2\beta_3) - (-1)^n \alpha_3(\alpha_4 - n^2\beta_n)\}/\delta_2$$

$$r_m = \{(\alpha_4 - \beta_4)\gamma_m^{(3)} + (\alpha_3 - \beta_3)\gamma_m^{(4)}\}/\delta_2$$

$$s_m = \{\alpha_4\gamma_m^{(3)} - \alpha_3\gamma_m^{(4)}\}/\delta_2$$

$$\delta_1 = 2\alpha_1\alpha_2 - \alpha_1\beta_2 + \alpha_2\beta_1 \neq 0\ ;$$

$$\delta_2 = 2\alpha_3\alpha_4 - \alpha_3\beta_4 + \alpha_4\beta_3 \neq 0.$$

Note here that the boundary conditions (22) and (23) are not all linearly independent; there exist four linear relations among them.

Equations (22) and (23) may be used to eliminate $\frac{1}{2}a_{0n}$, a_{1n}, $\frac{1}{2}a_{m0}$, and a_{m1} from the left-hand side of equation (19) to give

$$A_{m0}g_n + A_{m1}h_n + B_{0n}r_m + B_{1n}s_m + \sum_{i=0}^{M}(A_{mi} - \mu_i A_{m0} - \nu_i A_{m1})a_{in} + \sum_{i=2}^{N}(B_{jn} - U_j B_{0n} - V_j B_{1n})a_{mj} = \sum_{i=0}^{M}\ \sum_{j=0}^{N}{}'' A_{mi} f_{ij} B_{jn} \qquad m, n \geq 2$$

which may be written as

$$\sum_{i=2}^{M}(C_{m\,i}a_{in}) + \sum_{j=2}^{N} a_{m\,j}D_{jn} = b_{m\,n} \qquad m,n \geq 2 \tag{24}$$

where

$$\begin{aligned} C_{2i} &= A_{2i} - \mu_i A_{20} & D_{j2} &= B_{j2} - U_j B_{02} \\ C_{3i} &= A_{3i} - \nu_i A_{31} & D_{j3} &= B_{j3} - V_j B_{13} \\ C_{m\,i} &= A_{m\,i} \quad m \geq 4 & D_{jn} &= B_{jn} \quad n \geq 4 \end{aligned} \tag{25}$$

and

$$b_{m\,n} = \sum_{i=0}^{M}\ \sum_{j=0}^{N}{}'' A_{m\,i} f_{ij} B_{jn} - (A_{m\,0}g_n + A_{m\,1}h_n + B_{0\,n}r_m + B_{1\,n}s_m)$$

Further, let A be the $(M-1)\times(N-1)$ matrix with (m,n) element $a_{m+1,n+1}$, $1 \leq m \leq M-1, 1 \leq n \leq N-1$. Then equation (24) may be written in the matrix form

$$CA + AD = B \tag{26}$$

where C is an $(M-1)\times(M-1)$ matrix with (m,n) element $c_{m+1,n+1}$; D is an $(N-1)\times(N-1)$ matrix with (m,n) element $d_{m+1,n+1}$; and B is an $(M-1)\times(N-1)$ matrix with (m,n) element $b_{m+1,n+1}$.

It is worthy to note that the matrix C represents the Chebyshev approximation to $\partial^2/\partial x^2$ with the nonhomogeneous mixed boundary conditions, i.e. CA represents the first term in equation

(19) with the boundary conditions of equation (22) used to eliminate a_{0n} and a_{1n} for $0 \leq n \leq N$. The y-derivative in equation (19) appear above in the form AD.

The matrix equation (26) represents a system of linear equatons in the rectangular matrix A of the type $(M-1)(N-1)$ consisting of the elements a_{ij} $(2 \leq i \leq M, \ \ 2 \leq j \leq N)$. A method using the Kronecker matrix algebra for solving such system is to be considered in the next subsection.

3.2 Solution of the system of equations (26)

Let the Kronecker product of the two matrices C and D defined by

$$C \otimes D = [c_{ij} D] \qquad (i,j = 2,3,\dots,M)$$

and their Kronecker sum as

$$C \oplus D = C \otimes I_{N-1} + I_{M-1} \otimes D$$

where I_{M-1} and I_{N-1} are the identity matrices of order $(M-1)$ and $(N-1)$ respectively. Now, introducing the so called block vectors

$$\underline{a} = \begin{bmatrix} \underline{a}_2 \\ \underline{a}_3 \\ \vdots \\ \underline{a}_N \end{bmatrix} \qquad \underline{b} = \begin{bmatrix} \underline{b}_2 \\ \underline{b}_3 \\ \vdots \\ \underline{b}_N \end{bmatrix}$$

consisting of the columns of the matrices A and B respectively, where

$$A = [\underline{a}_2 \underline{a}_3 \ . \ . \ . \underline{a}_N] \ , \qquad B = [\underline{b}_2 \underline{b}_3 \ . \ . \ . \underline{b}_N]$$

then it is not difficult to show that the system (26) is equivalent to the following system

$$G\underline{a} = \underline{b} \tag{27}$$

where the coefficient matrix G is equal to the Kronecker sum $D^T \oplus C$, where D^T denotes the transpose of D. More detailed description of this algebra can be found in Graham (1981).

4. An alternative method of solution

If we assume that $u(x,y)$ and its partial derivatives $\partial^2 u/\partial x^2$ and $\partial^2 u/\partial y^2$ have bivariate Chebyshev series expansion given by (14) and (15), and make use of the recurrence relations of Chebyshev polynomials, it can easily be shown that

$$\left.\begin{aligned} a_{mn}^{(2,0)} &= \sideset{}{'}\sum_{\substack{i=m+2\\(i-m)\,\text{even}}}^{\infty} i(i^2-m^2)a_{in} \\ a_{mn}^{(0,2)} &= \sideset{}{'}\sum_{\substack{j=n+2\\(j-n)\,\text{even}}}^{\infty} j(j^2-n^2)a_{mj} \end{aligned}\right] \qquad (28)$$

It follows from (28) that the partial differential equation (11) - see equation (16) - is equivalent to

$$\sideset{}{'}\sum_{i=0}^{M} K_{mi}a_{in} + \sideset{}{'}\sum_{j=0}^{N} a_{mj}L_{jn} = f_{mn} \qquad m,n \geq 0 \qquad (29)$$

where

$$\begin{aligned} K_{mi} &= i(i^2-m^2) \qquad & i \geq m+2 \text{ and } (i-m) \text{ even} \\ L_{jn} &= j(j^2-n^2) \qquad & j \geq n+2 \text{ and } (j-n) \text{ even} \end{aligned}$$

and zero otherwise.

We propose now to assume that a_{in} and a_{mj} can be neglectged for $m \geq M+1$, $n \geq N+1$, and to eliminate $a_{i,N-1}$,

$a_{i,N}$, $a_{M-1,j}$ by making use of the boundary conditions (20) and (21). Now conditions (20) and (21) and after some lengthy manipulation, give

$$\left.\begin{aligned} a_{M-1,n} + \sum_{m=0}^{M-2\,\prime} \mu'_m a_{mn} &= g_n \\ a_{M,n} + \sum_{m=0}^{M-2\,\prime} \nu'_m a_{mn} &= h_n \end{aligned}\right] \quad n = 0, 1, \ldots, N. \tag{30}$$

$$\left.\begin{aligned} a_{m,N-1} + \sum_{n=0}^{N-2\,\prime} U'_n a_{mn} &= r_m \\ a_{m,N} + \sum_{n=0}^{M-2\,\prime} V'_n a_{mn} &= s_m \end{aligned}\right] \quad m = 0, 1, \ldots, M. \tag{31}$$

where

$$\begin{aligned}
\mu'_m &= \{(-1)^M (\alpha_1 - M^2\beta_1)(\alpha_2 + m^2\beta_2) - \\
&\qquad (-1)^m (\alpha_2 + M^2\beta_2)(\alpha_1 - m^2\beta_1)\} \,/\, \delta'_1 \\
\nu'_m &= \{(-1)^m (\alpha_1 - m^2\beta_1)(\alpha_2 + (M-1)^2\beta_2) + \\
&\qquad (-1)^M (\alpha_2 + m^2\beta_2)(\alpha_1 - (M-1)^2\beta_1)\} \,/\, \delta'_1 \\
g'_n &= \{(-1)^M (\alpha_1 - M^2\beta_1)\gamma_n^{(2)} - (\alpha_2 + M^2\beta_2)\gamma_n^{(1)}\} \,/\, \delta'_1 \\
h'_n &= \{(\alpha_2 + (M-1)^2\beta_2)\gamma_n^{(1)} + \\
&\qquad (-1)^M (\alpha_1 - (M-1)^2\beta_1)\gamma_n^{(2)}\} \,/\, \delta'_1 \\
U'_n &= \{(-1)^N (\alpha_3 - N^2\beta_3)(\alpha_4 + \beta_4) - \\
&\qquad (-1)^n (\alpha_4 + N^2\beta_4)(\alpha_3 - n^2\beta_3)\} \,/\, \delta'_2
\end{aligned}$$

$$
\begin{aligned}
V'_n &= \{(-1)^n(\alpha_3 - n^2\beta_3)(\alpha_4 + (N-1)^2\beta_4) + \\
&\qquad (-1)^N(\alpha_4 + n^2\beta_4)(\alpha_3 - (N-1)^2\beta_3)\} \;/\; \delta'_2 \\
r'_m &= \{(-1)^N(\alpha_3 - N^2\beta_3)\gamma_m^{(4)} - (\alpha_4 + N^2\beta_4)\gamma_m^{(3)}\} \;/\; \delta'_2 \\
s'_m &= \{(\alpha_4 + (N-1)^2\beta_4)\gamma_m^{(3)} + \\
&\qquad (-1)^N(\alpha_3 - (N-1)^2\beta_3)\gamma_m^{(4)}\} \;/\; \delta'_2 \\
\delta'_1 &= (-1)^M\{(\alpha_1 - (M-1)^2\beta_1)(\alpha_2 + M^2\beta_2) + \\
&\qquad (\alpha_1 + M^2\beta_1)(\alpha_2 - (M-1)^2\beta_2)\} \neq 0 \\
\delta'_2 &= (-1)^N\{(\alpha_3 - (M-1)^2\beta_3)(\alpha_4 + N^2\beta_4) + \\
&\qquad (\alpha_3 - N^2\beta_3)(\alpha_4 + (N-1)^2\beta_4)\} \neq 0
\end{aligned}
$$

Making use of (30) and (31) to eliminate $a_{M-1,m}$, $a_{M,n}$, $a_{m,N-1}$ and $a_{m,N}$ from the finite form of (29) leads to

$$
\sum_{i=0}^{M-2\;\prime} H_{mi}a_{in} + \sum_{j=0}^{N-2\;\prime} a_{mj}T_{jn} = b'_{mn} \qquad (32)
$$

$$
(0 \le m \le M-2 \;,\; 0 \le n \le N-2)
$$

where

$$
\begin{aligned}
H_{mi} &= K_{mi} - \mu'_i K_{m,M-1} - \nu'_i K_{m,M} \quad , \\
T_{jn} &= L_{jn} - U'_j L_{N-1,n} - V'_j L_{N,N} \quad , \\
b'_{mn} &= f_{mn} - (K_{m,M-1}g'_n + K_{m,M}h'_n + L_{N-1,n}r'_m + L_{N,n}s'_m).
\end{aligned}
$$

·Equation (32) may be written in the matrix form

$$
HA + AT = B' \qquad (33)
$$

where here A is the $(M-1)(N-1)$ matrix with (m,n) element a_{mn} , $0 \leq m \leq M-2$, $0 \leq n \leq N-2$. This equation has the same methods of solution like that of the matrix equation (26).

This alternative approach leads to an equation which is similar to equation (26). The approach of truncating the exact infinite Chebyshev expansion $u(x,y)$ by dropping the equations for the highest modes from equations (19) and (29) and determining them directly from the boundary conditions amounts to Lanczos' tau method. There is some computational advantages in this approach, in that equation (33) throws some light on the structure of the matrices H and T which were not previously apparent. Note that many of the elements of H and the transpose of T, including all those on and below the main diagonials are zero. Note also that although the method of this section is computationally simpler than that of Sec.3, it is matematically equivalent and will produce identical results.

It is worthy to mention here that the Chebyshev expansion methods of Sections 3 and 4 can be easily extended to handle Helmholtz equation

$$\nabla^2 u(x,y) + \lambda u(x,y) = f(x,y)$$

for constant λ subject to the most general boundary conditions (12) and (13) This needs a simple modification of (24) and (32) , by adding to their left and right hand sides, terms which reflect the steps of their derivations applied to the coefficients of $u(x,y)$.

Numerical results and comparisons

Consider the problem

$$\left.\begin{array}{l}\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=-32\pi^2\sin(4\pi x)\sin(4\pi y)\\ \text{subject to the boundary conditions}\\ u\pm\dfrac{\partial u}{\partial x}=\pm 4\pi\sin(4\pi y)\qquad x=\pm 1\\ u\pm\dfrac{\partial u}{\partial y}=\pm 4\pi\sin(4\pi x)\qquad y=\pm 1\end{array}\right] \tag{34}$$

This problem has the analytical solution

$$u(x,y)=\sin(4\pi x)\sin(4\pi y)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}{}'' a_{ij}T_i(x)T_j(y)$$

where

$$a_{ij}=\begin{cases}-4(-1)^{\frac{i+j}{2}}J_i(4\pi)J_j(4\pi) & i,j \text{ odd}\\ 0 & \text{otherwise}\end{cases}$$

and $J_i(z)$ denotes the Bessel function of the first kind. Note here that the coefficients a_{ij} decrease exponentially fast, because of the decrease of the Bessel functions as the order increase.

This model problem tests the Chebyshev approximation to Poisson's equation subject to mixed inhomogeneous boundary conditions which has a moderately oscillatory but otherwise well-behaved solution which is infinitely differentiable.

The proposed methods of Sections 3 and 4 are used to obtain the approximate solution

$$u(x,y) = \sum_{i=0}^{M} \sum_{j=0}^{N}{}'' a_{ij} T_i(x) T_j(y)$$

for the three cases $M = N = 16$, 24 32. The maximum absolute error of the Chebyshev approximation for each case is reported in Table 1. A comparison with the second order finite difference method for the two cases $M = N = 16$, 32, is illustrated in Table 2. For the well-behaved solution of this problem, the Chebyshev approximation is superior. This is clear by the improvement between $M = N = 16$, and $M = N = 32$; the error of the second-order scheme is reduced by a factor of 4, and that of the Chebyshev expansion by a factor of 10^9.

We next tested the Chebyshev approximation to the problem

$$\left.\begin{aligned} &\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \\ &\text{subject to the boundary conditions} \\ &u(x, \pm 1) = \cos(\frac{\pi x}{2}) \quad ; \quad u(\pm 1, y) = 0 \end{aligned}\right] \tag{35}$$

This problem is symmetric about both axes, and its analytical solution is given by

$$u(x,y) = \frac{\cos(\frac{\pi x}{2}) \cos h(\frac{\pi y}{2})}{\cos h(\frac{\pi}{2})} \quad .$$

The approximate solution with $M = N = 16$, is then

$$u(x,y) = \sum_{i=0}^{6} \sum_{j=0}^{6}{}'' a_{ij} T_i(x) T_j(y)$$

and the calculated non-zero values of the coefficients $\{a_{ij}\}$ are given in Table 3. The relations (7) can be used to evaluate $u(x,y)$

at any point (x, y). Comparison with the analytic solution at the points where $x = 0(0.2)1$, $y = 0(0.2)1$ shows a maximum difference of 10^{-5} between the approximate values and the analytic values. Relations (9) can be used to find the values of $|\nabla^2 u|$ at the same points. The maximum and minimum derivation from zero are 10^{-2} at (1,1) and 10^{-4} at (0,0). These values are calculated to provide some measure of the accuracy of the solutions.

With $M = N = 10$, (7) gives values of $u(x, y)$ at the points previously selected, in agreement to seven decimal places with the values from the analytical solution. The maximum derivation of $|\nabla^2 u|$ from zero is now less than 0.5×10^{-7}.

Now consider the problem

$$\left.\begin{array}{l} \dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} = 0 \\ \text{for which} \\ v(x,1) = v(1,y) = v(-1,y) = 0, \\ v(x,-1) = \overline{\gamma}(x) \ , \ \ \overline{\gamma}(\pm 1) \neq 0. \end{array}\right] \tag{36}$$

This problem has discontinuities at the two vertices $(\pm 1, -1)$. It is possible to deal with such singularities by observing that for any solution v of the homogeneous equation

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0,$$

derivatives of v are also solutions.Now, introduce the function $\gamma(x)$, such that

$$\frac{d^2 \gamma}{dx^2} + \overline{\gamma} = 0 \ , \ \ \gamma(\pm 1) = 0.$$

Then

$$v = \frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2}$$

where $u(x,y)$ is the solution of the problem

$$\begin{aligned} &\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \\ &u(x,1) = u(1,y) = u(-1,y) = 0, \\ &u(x,-1) = \gamma(x), \end{aligned} \tag{37}$$

in which there are no singularities at the verticies.

As an example take $\overline{\gamma}(x) = 1$, giving $\gamma(x) = \frac{1}{2}(1 - x^2)$ in (37). For that problem the required second derivatives can be calculated using (9) and (10). Thus the solution $v(x,y)$ of (36) is known because $v(x,y) = \partial^2 u \ / \ \partial x^2 = -\partial^2 u \ / \ \partial y^2$, where $u(x,y)$ is now the solution of (37). This problem is symmetric about the y-axis, and its analytical solution is given by

$$u(x,y) = -\frac{2}{\pi^3} \sum_{i=1}^{\infty} \frac{(-1)^i}{(i-\frac{1}{2})^3} \frac{\cos(i-\frac{1}{2})\pi x \sin h(i-\frac{1}{2})\pi(1-y)}{\sin h(2i-1)\pi}$$

The approximate solution with $M = N = 16$ is then

$$u(x,y) = \sum_{i=0}^{16} \sum_{j=0}^{16\,''} a_{ij} T_i(x) T_j(y)$$

where the coefficients $\{a_{ij}\}$ are given in Table 4. Using these coefficients with (7) to calculate $u(x,y)$ when $x = 0(0.2)1$, $y = -1(0.2 = 1$, and comparing with the corresponding values from the analytic solution, show that there is agreement up to seven decimal places at the selected points. Relations (9) can be used to calculate $|\nabla^2 u|$ at the same points, the values are almost zero at all internal points and differ from zero on the boundaries. Nevertheless, the solution $u(x,y)$ does fit the boundary conditions very accurately.

It is worthy to note that if $M = N = 10$, then the coefficients $\{a_{ij}\}$ in this solution differ slightly from those obtained with $M = N = 16$, and the values of $u(x,y)$ obtained from this economised solution differ from those with $M = N = 16$, by 2×10^{-6} at most, at the previously selected points.

From the result of the last two problems it is seen that although a greater number of terms is needed to produce more accurate results, the solution from fewer terms is also adequate even if values of $|\nabla^2 u|$ are not very satisfactory. It appears that accurate solutions may be obtained with a smaller number of terms but that if corresponding accuracy is needed for derivative values, then a larger number of terms will be required.

Consider the problem

$$\left.\begin{aligned} &\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - 3u = 1 \\ &u(x,\pm 1) = 0 \quad , \quad u(\pm 1, y) = 0 \end{aligned}\right] \tag{38}$$

With $M = N = 20$, the coefficients in the solution (14) are less than 10^{-6} in magnitude for $m,n \geq 16$ and lead to the numerical results given in Table 5. This example serves to illustrate the application of the proposed methods for solving Helmholtz equation.

The numerical results for the previous examples were obtained by solving the algebraic equations for the series coefficients using the iterative method of successive over-relaxation; values of the over-relaxation parameter of between 1.8 and 1.95 were found most effective in keeping down the number of iterations. Convergence was assumed when the relative error of two successive iterates is less than 10^9, for every coefficient a_{ij} in the solution.

To end this section, it is worth to be mentioned that this type of doubly-Chebyshev spectral approximation for elliptic equations can be greatly extended and generalized to parabolic equations in two-space variables with constant or even variable coefficients. It can also be extended and generalized by using collocation (pseudo-spectral) method to find the expansion coefficients implicitly, instead of matching two series expansions. Examples of this approach for parabolic and elliptic equations in one-space variable are given in Pasciak (1980), Gottlieb (1981) and Funaro (1988).

6. Some concluding remarks

As both the general theory of the convergence of spectral methods suggests and the first problem (34) illustrates, whenever the solution $u(x,y)$ is infinitely differentiable, the Chebyshev expansion can achieve highly accurate solutions to Poisson's equation far more efficiently than the standard finite-difference methods. Note also that the Chebyshev expansion method requires less demands of computer storage since fewer degrees of freedom are needed.

Spectral methods for Poisson's equation, of course, need not be based on Chebyshev polynomials. In some applications an expansion in Legendre polynomials may be more appropriate.

The use of the ultraspherical polynomials spectral approximation has been recently applied by the author (1989) to the third boundary value problem for parabolic equation in one-space variable. Use of doubly-ultraspherical spectral methods to parabolic and elliptic equations in two-space variables can also be extended and generalized easily. We hope to publish this in a forthcoming paper. Note here that Chebyshev and Legendre expansions may be considered as special cases from the ultraspherical expansion.

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TABLE 1

Maximum absolute error of the spectral approximations to problem (34) as a function of the number of degrees of freedom M and N.

M=N	Maximum absolute error
16	3.3×10^{-2}
24	6.9×10^{-6}
32	4.8×10^{-11}

TABLE 2

Maximum absolute error of the second-order finite difference (SOFD), and the spectral Chebyshev apoximations (SCA), to problem (34) as a function of degrees of freedom M and N.

M=N	SOFD	SCA
16	2.3×10^{-1}	3.3×10^{-2}
32	5.3×10^{-2}	4.8×10^{-11}

TABLE 3

Non-zero coefficients ($\times 10^6$) in the approximate solution of problem (35).

i \ j	0	2	4	6
0	1293257	283627	013479	268
2	-684173	-150045	-7131	-142
4	38348	8 410	400	8
6	-817	-179	-9	-0

TABLE 4

Non-zero coefficients ($\times 10^6$) in the approximate solution of problem (37), with $M = N = 16$.

$i \backslash j$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
0	338236	-227157	76141	-21691	4418	-1028	267	-96	41	-20	10	-6	3	-2	1	-1	1
2	-173859	115620	-36958	9442	-1256	21	99	-59	31	-16	9	-5	3	-2	1	-1	1
4	4418	-1757	-1256	1374	-831	358	-124	33	-5	-3	3	-3	2	-1	1	-1	0
6	267	-231	99	55	-124	119	-82	47	-23	10	-4	1	0	0	0	0	0
8	41	-39	31	-14	-5	18	-23	21	-16	10	-6	3	-2	1	0	0	0
10	10	-10	9	-7	3	1	-4	6	-6	5	-4	3	-2	1	-1	0	0
12	3	-3	3	-3	2	-1	0	1	-2	2	-2	2	-1	1	-1	1	0
14	1	-1	1	-1	1	-1	0	0	0	1	-1	1	-1	1	-1	0	0
16	1	-1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0

TABLE 5

Values of $u(x, y)$, $x, y = 0(0.2)1, M = N = 20$, for problem (38).

$i \backslash j$	0.00	0.20	0.40	0.60	0.80	1.00
0.00	-0.17466	-0.16982	-0.15431	-0.12506	-0.07650	0.00000
0.20	-0.16982	-0.16517	-0.15027	-0.12203	-0.07486	0.00000
0.40	-0.15431	-0.15027	-0.13721	-0.11216	-0.06950	0.00000
0.60	-0.12506	-0.12203	-0.11216	-0.09286	-0.05876	0.00000
0.80	-0.07650	-0.07486	-0.06950	-0.05876	-0.03880	0.00000
1.00	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000