*Annales Univ. Sci. Budapest., Sect. Comp.* 10 (1989) 211-220

# ON A FREE BOUNDARY TWO-DIMESIONAL PROBLEM

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1. The problem is summarized as follows: Let $u$ satisfies the elliptic differential equation

(1)
$$Au := -\frac{\partial}{\partial r}\left(\frac{1}{r^3}\frac{\partial u}{\partial r}\right) - \frac{\partial}{\partial z}\left(\frac{1}{r^3}\frac{\partial u}{\partial z}\right) = 0,$$

in the free boundary domain $\Omega_e \subset \Omega$ (Fig.1) together with the boundary conditions:

(2)
$$u(z,0) = 0,$$

(3)
$$\frac{\partial u(z,r)}{\partial z}\Big|_{z=0} = 0, \quad \frac{\partial u(z,r)}{\partial z}\Big|_{z=Z} = 0$$

and across $R = const$, the following conditions have to be satisfied:

(4)
$$u(z,R) = u^p(z,R) = f(z),$$

(5)
$$\nabla u(z,r)\,|_{r=R} = \nabla u^p(z,r)\,|_{r=R},$$

where $u^p \in W_2^2$ is a given function in the domain $\Omega$.

The aim of this work is to use the analytic representation for the solution of equation (1) to propose an iteration process for the above mentioned free boundary problem.

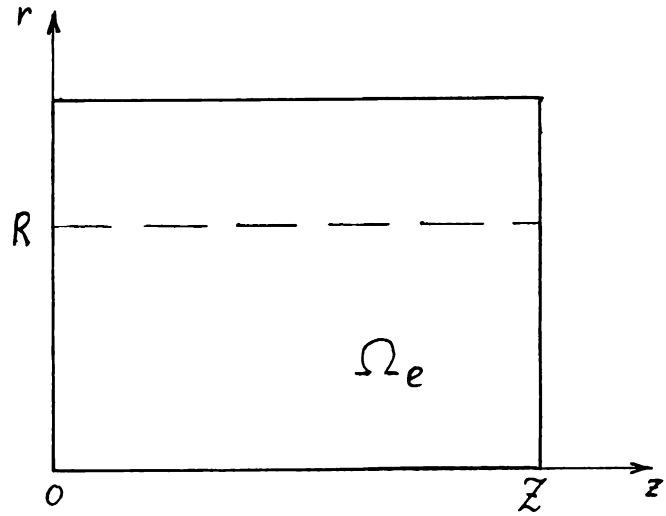


Fig. 1. The free boundary domain $\Omega_e$

2. The solution of equation (1) in the given domain $\Omega_e$ with the given boundary condition $f(z)$ on $r = R$ was obtained by spline function method in [1]. Now we are going to find the analytical solution for the same problem by using infinit series.

Suppose $u(z,r) = Z^*(z)R^*(r)$ and substituting it in (1), we are led to two ordinary diferential equations for $Z^*$ and $R^*$

(6)
$$\frac{d^2Z^*}{dz^2} + \lambda Z^* = 0,$$

(7)
$$\frac{d^2R^*}{dr^2} - \frac{3}{r}\frac{dR^*}{dr} - \lambda R^* = 0,$$

where $\lambda$ is arbitrary coefficient. The boundary conditions for $Z^*$ and $R^*$ can be obtained from (2) -(4).

If $\lambda = 0$, the solutions of (6) and (7) have the following forms:

(8)
$$Z^* = const, \qquad R^* = b_0 r^4.$$

If $\lambda \neq 0$ the solution of the eigenvalue problem (6) with the boundary conditions (3) has the following form

(9)
$$Z^* = c_k cos(\sqrt{\lambda_k} z),$$

where $\lambda_k = \frac{\pi^2 k^2}{Z^2}$, $k = 1, 2, \ldots$

It is known that the solution of the modified Bessel's equation

$$\frac{d^2 w}{dz^2} + \frac{1 - 2a}{z}\frac{dw}{dz} + \left((bcz^{c-1})^2 + \frac{a^2 - m^2 c^2}{z^2}\right)w = 0,$$

with the boundary condition (2) can be written as follows [2]

(10)
$$w = z^a J_m(bz^c)$$

where $J_m$ is the first kind Bessel's function. The solution of the equation (7) can be obtained from (10) if we let $a = 2, c = 1$, $b^2 = -\lambda_k, m = 2$. Therefore

(11)
$$R^* = Cr^2 I_2(\sqrt{\lambda_k} r),$$

where $I_2$ is the modified Bessel's function.

Using the Fourier series for (8), (9) and (11) we can get the solution of equation (1) with conditions (2) - (3) as

(12)
$$u(z, r) = c_0 r^4 + r^2 \sum_{k=1}^{\infty} c_k cos\frac{\pi k z}{Z} I_2\left(\frac{\pi k r}{Z}\right).$$

The function $f(z)$, which is supposed to satisfy the boundary conditions $f'(0) = f'(Z) = 0$, can be expressed by the Fourier series as following

(13)
$$f(z) = a_0 + \sum_{k=1}^{\infty} a_k cos\frac{\pi k z}{Z},$$

where the Fourier coefficients of $f(z)$ have the following forms:
(14)
$$a_0 = \frac{1}{Z}\int_0^Z f(z)dz, \quad a_k = \frac{2}{Z}\int_0^Z f(z)cos\frac{\pi kz}{Z}dz.$$

Let us note that $f''(z) = -\frac{\pi^2}{Z^2}\sum_{k=1}^{\infty} a_k k^2 cos\frac{\pi kz}{Z}$, and because $u^p \in W_2^2(\Omega)$, we have that $f''(z) \in L_2$. Hence for the Fourier coefficients we can write the following estimate [2]
(15)
$$| a_k k^2 | \underset{k \to 0}{\to} 0.$$

To obtain the values of the coefficients $c_0$ and $c_k$, let us apply condition (4), $u(z,R) = f(z)$, and so we get
(16)
$$c_0 = \frac{a^0}{R^4}, c_k = \frac{a_k}{R^2}(I_2(\frac{\pi kR}{Z}))^{-1}, k = 1,2,\ldots$$

By substituting (16) into (12) we have
(17)
$$u(z,r) = a_0\frac{r^4}{R^4} + \frac{r^2}{R^2}\sum_{k=1}^{\infty} a_k cos\frac{\pi kz}{Z} I_2(\frac{\pi kr}{Z})(I_2(\frac{\pi kR}{Z}))^{-1}.$$

3. To construct an iterative process for obtaining the free boundary let $\tilde{R}$ be the approximate free boundary of $\Omega_e$:
(18)
$$\tilde{R} = R + \delta, \quad \tilde{R} = const,$$

where $R$ represents the unknown exact boundary of $\Omega_e$, and $\delta$ is supposed to be small. We assumed that $u^p$ is given function in $\Omega$, therefore on $\tilde{R}$ the function

(19)
$$\tilde{f}(z) = u^p(z, \tilde{R})$$

is known.

Similarly as in (13) we can write

(20)
$$\tilde{f}(z) = \tilde{a}_0 + \sum_{k=1}^{\infty} \tilde{a}_k \cos\frac{\pi k z}{Z}.$$

In the domain $\tilde{\Omega}_e$ with the boundary $\tilde{R}$ the solution of (1), denoted by $\tilde{u}(z,r)$, with the boundary function (19) can be written in the form

(21)
$$\tilde{u}(z,r) = a_0\frac{r^4}{\tilde{R}^4} + \frac{r^2}{\tilde{R}^2}\sum_{k=1}^{\infty} \tilde{a}_k \cos\frac{\pi k z}{Z} I_2\left(\frac{\pi k r}{Z}\right)\left(I_2\left(\frac{\pi k \tilde{R}}{Z}\right)\right)^{-1}.$$

Let us write the relations between $a_k$ and $\tilde{a}_k$. Using the Taylor's expansions and (18) the following can be obtained for (19)

(22)
$$\tilde{f}(z) = u^p(z,R) + \delta\frac{\partial u^p}{\partial r}(z,R) + 0(\delta^2)$$

Hence from (14) and (22) we obtain

(23)
$$\tilde{a}_0 = a_0 + \delta\frac{1}{Z}\int_0^Z \frac{\partial u^p}{\partial r}\Big|_R \, dz + 0(\delta^2),$$

and similarly for $\tilde{a}_k, k = 1, 2, \dots$ we have

$$\tilde{a}_k = a_k + \delta\frac{2}{Z}\int_0^Z \frac{\partial u^p}{\partial r}\Big|_R \cos\frac{\pi k z}{Z} dz + 0(\delta^2).$$

Now from the formulation of the problem, it is obvious that

(25)
$$\tilde{u}(z,\tilde{R}) = u^p(z,\tilde{R}) \equiv \tilde{f}(z),$$
$$\frac{\partial \tilde{u}}{\partial z}(z,\tilde{R}) = \frac{\partial u^p}{\partial z}(z,\tilde{R}) \equiv \frac{d\tilde{f}}{dz}(z).$$

But for $\tilde{R} \neq R$

(26)
$$\frac{\partial \tilde{u}}{\partial r}(z,\tilde{R}) \neq \frac{\partial u^p}{\partial r}(z,\tilde{R}).$$

This means that the second condition in (5) is not satisfied. We shall use this fact to determine, from the difference between $\partial\tilde{u}/\partial r$ and $\partial u^p/\partial r$ on $r = \tilde{R}$, the approximate value of $\delta$.

From (21) let us obtain $\partial\tilde{u}/\partial r$ on the line $r = \tilde{R}$

(27)
$$\frac{\partial \tilde{u}}{\partial r}\Big|_{r=\tilde{R}} = \frac{4\tilde{a}_0}{\tilde{R}} + \frac{2}{\tilde{R}}\sum_{k=1}^{\infty} \tilde{a}_k \cos\frac{\pi k z}{Z} +$$
$$+\sum_{k=1}^{\infty} \tilde{a}_k \cos\frac{\pi k z}{Z} I_2'\left(\frac{\pi k \tilde{R}}{Z}\right)\left(I_2\left(\frac{\pi k \tilde{R}}{Z}\right)\right)^{-1}\frac{\pi k}{Z},$$

where $I_2'$ denotes the first derivative over the argument. By substituting (18), (23), (24) into (27) and using that

$$\frac{I_2'\left(\frac{\pi k(R+\delta)}{Z}\right)}{I_2\left(\frac{\pi k(R+\delta)}{Z}\right)} = \frac{I_2'\left(\frac{\pi k R}{Z}\right)}{I_2\left(\frac{\pi k R}{Z}\right)} + \frac{\pi k}{Z}\delta\left(\frac{I_2'\left(\frac{\pi k R}{Z}\right)}{I_2\left(\frac{\pi k R}{Z}\right)}\right)_R' + 0(\delta^2),$$

we get with an arrangement to the terms

$$\frac{\partial \tilde{u}}{\partial r}\mid_{r=\tilde{R}}= \frac{4a_0}{R}+\frac{2}{R}\sum_{k=1}^{\infty} a_k cos\frac{\pi kz}{Z}+\sum_{k=1}^{\infty} a_k\frac{\pi k}{Z}cos\frac{\pi kz}{Z}I_2'(\frac{\pi kR}{Z})*$$

$$*(I_2(\frac{\pi kR}{Z}))^{-1}+\delta[-\frac{2a_0}{R^2}-\frac{2}{R^2}(a_0+\sum_{k=1}^{\infty} a_k cos\frac{\pi kz}{Z})+\frac{2}{RZ}*$$

(28)

$$*\int_0^Z \frac{\partial u^p}{\partial r}\mid_R dz+\frac{2}{R}(\frac{1}{Z}\int_0^Z \frac{\partial u^p}{\partial r}\mid_R dz+\sum_{k=1}^{\infty}$$

$$(\frac{2}{Z}\int_0^Z \frac{\partial u^p}{\partial r}\mid_R cos\frac{\pi kz}{Z}dz)*$$

$$*cos\frac{\pi kz}{Z})+\sum_{k=1}^{\infty}(\frac{2}{Z}\int_0^Z \frac{\partial u^p}{\partial r}\mid_R cos\frac{\pi kz}{Z}dz)\frac{\pi k}{Z}cos\frac{\pi kz}{Z}I_2'(\frac{\pi kR}{Z})*$$

$$*(I_2(\frac{\pi kR}{Z}))^{-1}+\sum_{k=1}^{\infty} a_k(\frac{\pi k}{Z})^2 cos\frac{\pi kz}{Z}(\frac{I_2'(\frac{\pi kR}{Z})}{I_2(\frac{\pi kR}{Z})})_R']+0(\delta^2),$$

To simplify (28) let us assume that

(28)

$$\frac{\partial u^p}{\partial r}\mid_R= g_0+\sum_{k=1}^{\infty} g_k cos\frac{\pi kz}{Z},$$

where

$$g_0=\frac{1}{Z}\int_0^Z \frac{\partial u^p}{\partial r}\mid_R dz$$

$$g_k = \frac{2}{Z}\int_0^Z \frac{\partial u^p}{\partial r}\Big|_R \cos\frac{\pi k z}{Z} dz, k = 1,2,\ldots$$

Now one can see that from (28) the difference between $\partial\tilde{u}/\partial r$ and $\partial u^p/\partial r$ on $r = \tilde{R}$ can be determined as a power series in a small parameter $\delta$. The coefficients of the first term of the series can be expressed by the values of the known functions on the boundary $r = \tilde{R}$ as following

(30)

$$\frac{\partial\tilde{u}}{\partial r}\Big|_{\tilde{R}} = \delta\Big[ -\frac{\partial^2 u^p}{\partial r^2}\Big|_{\tilde{R}} - \frac{2\tilde{a}_0}{\tilde{R}^2} - \frac{1}{\tilde{R}^2}u^p(z,\tilde{R}) + \frac{2}{\tilde{R}}\tilde{g}_0 +$$

$$+\frac{2}{\tilde{R}}\frac{\partial u^p}{\partial r}\Big|_{\tilde{R}} + \sum_{k=1}^{\infty}\tilde{g}_k\frac{\pi k}{Z}\cos\frac{\pi k z}{Z} I_2'\Big(\frac{\pi k\tilde{R}}{Z}\Big)\Big(I_2\Big(\frac{\pi k\tilde{R}}{Z}\Big)\Big)^{-1} +$$

$$+\sum_{k=1}^{\infty}\tilde{a}_k\Big(\frac{\pi k}{Z}\Big)^2\cos\frac{\pi k z}{Z}\Big(\frac{I_2'\big(\frac{\pi k\tilde{R}}{Z}\big)}{I_2\big(\frac{\pi k\tilde{R}}{Z}\big)}\Big)'\Big] + 0(\delta^2).$$

where $\tilde{g}_k$ are the Fourier coefficients of $\partial u^p/\partial r$ on $\tilde{R}$.

Thus the boundness of the first five terms in the right hand side of (30) are obvious. Let us consider the estimations of the last two series in (30).

Because of $u^p \in W_2^2$, we can suppose in addition that $\partial^3 u^p/\partial z^2\partial r \in L_2$. Therefore, as that obtained in (15), for $\tilde{g}_k$ the following estimation is true

(31)

$$|\tilde{g}_k k^2| \underset{k\to\infty}{\longrightarrow} 0,$$

and obviously $\tilde{g}_k k$ tends to zero more fastly than $k^{-1}$.

It can be shown that for $I_2$ the following asymptotic expressions, for a big argument $y$, are true

$$\frac{I_2'(y)}{I_2(y)} = 1 - \frac{1}{2y} + 0(y^{-2}),$$

(32)

$$\left(\frac{I_2'(y)}{I_2(y)}\right)' = \frac{1}{2y^2} + 0(y^{-3}).$$

From the relations (15), (31) and (32) the convergence of the last two series in (30) are followed. The numerical computations have shown that the first expression in (32) is less than 1 starting with $y = 5$. The asolute value of the product of the second expression in (32) with $y^2$ is less than 1 starting with $y = 4$.

It is possible that both sides of (30) are functions of $z$, while $\delta$ is supposed to be constant. To consider this case, let us integrate the two sides of (30) on the interval $(0, Z)$, and so we get

(33)

$$\frac{1}{Z}\int_0^Z \frac{\partial \tilde{u}}{\partial r}\Big|_{\tilde{R}}\, dz - \tilde{g}_0 = \delta\Big[-\frac{1}{Z}\int_0^Z \frac{\partial^2 u^p}{\partial r^2}\Big|_{\tilde{R}}\, dz - \frac{4\tilde{a}_0}{\tilde{R}^2} - \frac{4}{\tilde{R}}\tilde{g}_0\Big] + 0(\delta^2).$$

Clearly, it can be seen that the right side of (33) have constant terms. Therefore the approximate value of $\delta$ can be obtained. Thus formulas (30) and (33) can be applied as the base of iteration process for solving the formulated free boundary problem.

## References

[ 1 ] **Mohamed, A.Sh.:** Approximate Solution for an Elastic Problem by Spline Function. Annales Univ. Sci. Budapest, Sectio Computat. 10, 1989.

[ 2 ] **Korn, G.A. - Korn, T. M.:** Mathematical Hand-Book For Scientists and Engineers. McGrawhill 1961.