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APPROXIMATE SOLUTION FOR AN ELASTIC PROBLEM BY SPLINE FUNCTION

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1. Introduction

The elastic problems have many types and are formulated in different forms. In this work we shall consider an elastic axisymmetric cylinder subjected to torque T applied at both ends (see FIG. 1.1). Following the problem formulation introduced by C.W. Gryer (1), we shall give an approximate solution to the problem by using spline functions together with the more general boundary conditions.

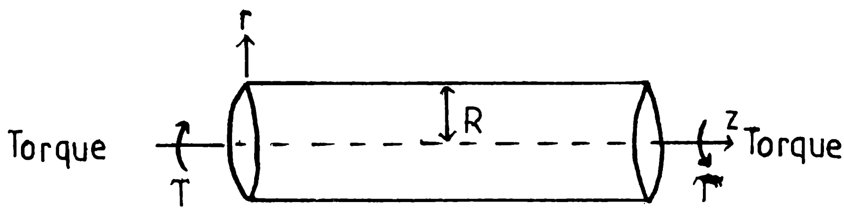

FIGURE 1.1. An axisymmetric cylinder

The problem formulation (See (1)) can be in a two dimensional domain Ω in the rz-plane because of the axial symmetry (FIG. 1.2).

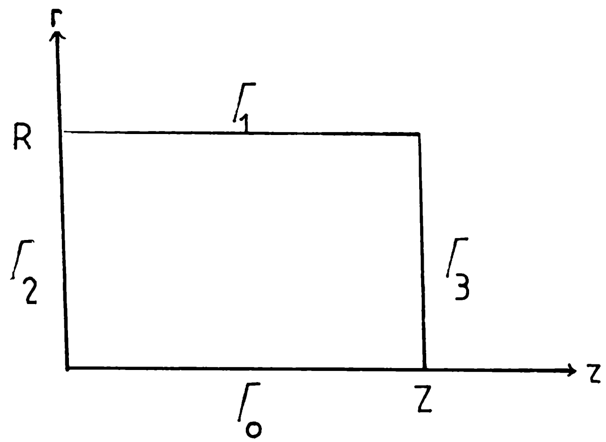

FIGURE 1.2. The domain Ω

The problem reduced to find function u which must satisfy the elliptic differential equation:

(1.1)
$$Au = -\frac{\partial}{\partial r}\left(\frac{1}{r^3}\frac{\partial u}{\partial r}\right) - \frac{\partial}{\partial z}\left(\frac{1}{r^3}\frac{\partial u}{\partial z}\right) = 0; \quad \text{in } \Omega;$$

together with the boundary conditions:

(1.2)
$$u = 0; \qquad \text{on } \Gamma_0;$$

(1.3)
$$u = f_2(z); \qquad \text{on } \Gamma_1;$$

(1.4)
$$\frac{\partial u}{\partial n} = -\frac{\partial u}{\partial z} = \varphi_1(r); \quad \text{on } \Gamma_2;$$

(1.5)
$$\frac{\partial u}{\partial n} = \frac{\partial u}{\partial z} = \varphi_2(r); \quad \text{on } \Gamma_3;$$

where $\Gamma_0, \Gamma_1, \Gamma_2$ and Γ_3 are as shown in Figure 1.2.

Remark 1. Aforementioned formulation assumes the material state of the problem is only elastic, that is the increasing of the stress function $q = r^{-2}(\nabla u)$ is limited by a constant k (k can be determined from the material property) in order that the plastic region would not be formulated. We shall consider the elastic-plastic problem in our forthcoming work.

In this paper we intend to discuss the construction and the properties of the spline functions, as well as using Ritz method to obtain the approximate solution of (1.1), and proving its convergence.

2. The spline functions construction.

The domain Ω (FIG. 1.2) is divided into (nxm) rectangular subdomains by lines parallel to the rectangular coordinates.

Let

(2.1)
$$\Omega \cup \Gamma_0 \cup \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 = \cup \bar{G}_{i,j}; \quad i = \overline{0, n-1};$$
$$j = \overline{0, m-1}; n \geq 1; m \geq 2;$$

where

$$\bar{G}_{i,j} = \{z, r \mid z_i \leq z \leq z_{i+1}; r_j \leq r \leq r_{j+1}; z_i = i \cdot h_z;$$
$$r_j = j \cdot h_r; z_0 = 0, z_n = Z; r_0 = 0, r_m = R\}.$$

Let the solution of (1.1) be approximated by the spline function:

(2.2)
$$u(z, r) \approx S_\Delta(z, r) := S_{i,j}(z, r); \quad \text{in } \bar{G}_{i,j}.$$

In constructiong $S_\Delta(z, r)$ the following steps have been considered

(i) $S_\Delta(z,r)$ in every subdomain is identical with a polynomial of two variables.

(ii) Let $S_\Delta(z,r)$ satisfy (1.1) in the interior of the subdomains

(2.3)
$$AS_{i,j}(z,r)=0; \quad \text{in } \; G_{i,j}.$$

(iii) Depending upon the result of (2.3) we can choose a minimal degree polynomial of two variables that satisfy (2.3) nontrivially, therefore, we get

(2.4)
$$S_{i,j}(z,r)=A_{i,j}r^4+B_{i,j}r^4z+C_{ij}z+D_{i,j};$$

where $A_{i,j}, B_{i,j}, C_{i,j}$ and $D_{i,j}$ are arbitrary constants.

(iv) The spline functions (2.4) are supposed to be continuous in Ω.

Remark 2. From formula (2.4) we can obtain the exact solution of (1.1) for the one and two dimensional problems when $\varphi_1(r)=\varphi_2(r)=0$ and $f_2=$ constant. Thus $u=f_2r^4/R^4$.

Hence, we can suppose that (2.4) gives a better approximate solution to (1.1) with the more general conditions (1.2 - 1.5), than a piece wise linear approximation.

Let us rewrite formula (2.4) in the analogous form:

(2.5)
$$S_{i,j}(z,r)=a_{i,j}(r^4-r_j^4)+b_{i,j}(r^4-r_j^4)(z-z_i)+$$
$$+c_{i,j}(z-z_i)+d_{i,j}; i=\overline{0,n-1}; j=\overline{0,m-1};$$

where $a_{i,j}, b_{i,j}, c_{i,j}$ and $d_{i,j}$ are arbitrary constants.

We suppose that (2.5) will only satisfy the main boundary conditions (1.2) and (1.3) but not by all means the natural boundary conditions (1.4) and (1.5) [2].

Therefore (2.5) shall have the following forms in the first and last rows in $\bar{\Omega}$:

(2.6)

$$S_{i,0}(z,r) = a_{i,0}r^4 + b_{i,0}r^4(z - z_i); \quad i = \overline{0, n-1};$$

(2.7)

$$S_{i,m-1}(z,r) = a_{i,m-1}(r^4 - R^4) + b_{i,m-1}(r^4 - R^4)(z - z_i) + f_2(z);$$

$$i = \overline{0, n-1}.$$

For the other rows in $\bar{\Omega}$ (i.e. $j = \overline{1, m-2}$) the spline functions shall have the form (2.5).

THEOREM 1. The continuous spline functions (2.5) exist in $\bar{\Omega}$ and satisfy the boundary conditions (1.2) and (1.3).

PROOF. The conditions that the spline functions (2.5-2.7) be continuous in $\bar{\Omega}$, and take on the main boundary values, are expressed by the following system of equations

(2.8)

$$a_{i,j} + b_{i,j}(z_{i+1} - z_i) - a_{i+1,j} = 0; \quad i = \overline{0, n-2}, j = \overline{0, m-1}.$$

(2.9)

$$c_{i,j}(z_{i+1} - z_i) + d_{i,j} - d_{i+1,j} = 0; \quad i = \overline{0, n-2}, j = \overline{1, m-2}.$$

(2.10)

$$a_{i,0}r_1^4 - d_{i,1} = 0; \qquad i = \overline{0, n-1}.$$

(2.11)

$$b_{i,0}r_1^4 - c_{i,1} = 0; \qquad i = \overline{0, n-1}.$$

(2.12)

$$a_{i,j}(r_{j+1}^4 - r_j^4) + d_{i,j} - d_{i,j+1} = 0; \quad i = \overline{0, n-1}, j = \overline{1, m-3}.$$

(2.13)

$$b_{i,j}(r_{j+1}^4 - r_j^4) + c_{i,j} - c_{i,j+1} = 0; i = \overline{0, n-1}, j = \overline{1, m-3}.$$

(2.14)

$$a_{i,m-2}(r_{m-1}^4 - r_{m-2}^4) + d_{i,m-2} - a_{i,m-1}(r_{m-1}^4 - r_m^4) =$$
$$= f_2(z_i), \quad i = \overline{0, n-1}.$$

(2.15)

$$b_{i,m-2}(r_{m-1}^4 - r_{m-2}^4) + c_{i,m-2} - b_{i,m-1}(r_{m-1}^4 - r_m^4) =$$
$$= \frac{f_2(z_{i+1}) - f_2(z_i)}{z_{i+1} - z_i}; \quad i = \overline{0, n-1}.$$

The system (2.8-2.15) has 2(2mn-m-2n+1) equations and 4(mn-n) coefficients. The analysis shows that there exists (mn-nm+1) dependent equations in the system. By deleting the dependent equations from the system (using the equations (2.10) (2.12) and (2.14) for i=0 only) we get the system $\bar{M}$ of $\bar{N}$ = (3mn-3n-m+1) independent equations and 4(mn-n) unknown coefficients. Then by induction we get that the rang of matrix P(P is the coefficient matrix of the system $\bar{M}$) is equal to the rang of matrix $\bar{P}$($\bar{P}$ is obtained by adjoining to marix P the column made up of the right hand side terms of the system $\bar{M}$). Therefore the system $\bar{M}$ is consistent (3) and its solution exists but not unique, hence, we have now a family of continuous splines in Ω.

3. The spline functions properties.

We have a space of continuous spline functions, where these splines are statisfing the main boundary conditions and contains (mn-n+m-1) free coefficients. Ritz method has been chosen to obtain the solution of (1.1) together with the natural boundary conditions (1.4) and (1.5).

By reformulating the original problem equivalently, we can see that a generalized solution of (1.1) exists in a Hilbert space. To prove this let

(3.1)
$$u = \bar{u} + \tilde{u}; \qquad \tilde{u} = \frac{r^4}{R^4} f_2(z);$$

therefore,

(3.2)
$$Au = A\bar{u} = 0.$$

From (1.1)-(1.5), (3.1) and (3.2), we obtain the following non-homogenous problem for $\bar{u}$

(3.3)
$$A\bar{u} = F := \frac{r}{R^4}\frac{d^2 f_2(z)}{dz^2};$$

(3.4)
$$\bar{u}(z,0) = \bar{u}(z,R) = 0,$$

(3.5)
$$\frac{\partial \bar{u}}{\partial z}(0,r) = -\bar{\varphi}_1(r) := -\varphi_1(r) - \frac{r^4}{R^4}\frac{df_2(0)}{dz},$$

(3.6)
$$\frac{\partial \bar{u}}{\partial z}(Z,r) = -\bar{\varphi}_2(r) := \varphi_2(r) - \frac{r^4}{R^4}\frac{df_2(Z)}{dz},$$

which is equivalent to (1.1)-(1.5).

Now the domain of definition of the operator A(1.1) is

(3.7)
$$D(A) = \{v \mid Av \in L_2(\Omega), v(z,0) = v(z,R) = 0\}.$$

Let us define the inner product [2,4,5]

(3.8)
$$[u,v] = \int\limits_{\Omega} \frac{1}{r^3}\left[\frac{\partial u}{\partial r}\frac{\partial v}{\partial r} + \frac{\partial u}{\partial z}\frac{\partial v}{\partial z}\right]dzdr,$$

where (3.8) is bilinear finctional and has all the properties of the inner product in the Hilbert space, therefore we can define the energetical Hilbert space H_A (H_A here also denotes the weighted Sobolev space (1)):

(3.9)
$$H_A = \{v \mid \frac{1}{r^{3(2}}((\frac{\partial v}{\partial r})^2 + (\frac{\partial v}{\partial z})^2)^{1/2} \in L_2(\Omega),$$
$$v(z,0) = v(z,R) = 0\}.$$

If $u, v \in H_A$, then the product (3.8) make sense [2,4,5].

Let the norm of v in H_A be as follows

(3.10)
$$[v] = [v,v]^{1/2}.$$

Now if for every $v \in H_A$, $\bar{u}$ satisfies the following equality

(3.11)
$$[\bar{u},v] = (\bar{F},v) := -\int\limits_{\Omega} \frac{r}{R^4}\frac{df_2}{dz}\frac{\partial v}{\partial z}dzdr +$$
$$+\int\limits_0^R \frac{1}{R^3}Q_1(r)v(0,r)dr + \int\limits_0^R \frac{1}{R^3}Q_2(r)v(Z,r)dr;$$

then $\bar{u}$ is the generalized solution of (3.3)-(3.6) [2.4].

It can be seen that the operator $(\bar{F},v)$ is bounded in H_A

(3.12)
$$\mid (\bar{F},v) \mid \leq c[v],$$

if the following are satisfied

$(i) f_2(z) \in w_2^1[0,Z]$,

$(ii) \frac{1}{r^{3/2}} Q_k(r) \in L_2[0, R], k = 1, 2.$

Therefore, the generalized solution $\bar{u}$ of (3.3) exists in the Hilbert space H_A (2), and according to that the generalized solution $u = \bar{u} + \tilde{u}$ of (1.1) exists too.

The approximation of the generalized solution is sought in a set of splines by the Ritz method, and from the variational method theory (2.5), $\bar{u}$ is minimizing the functional $J(v)$:

(3.13)

$$J(v) = [v, v] - 2(\bar{F}, v).$$

To use the continuous conditions of the set of the splines S_Δ, we rewrite the form of the functional (3.13):

(3.14)

$$J(S_\Delta) = [S_\Delta, S_\Delta] - 2(\bar{F}, S_\Delta) + \sum_{\bar{k}=1}^{\bar{N}} \lambda_{\bar{k}} g_{\bar{k}},$$

where $\lambda_{\bar{k}}\ (\bar{k} = \overline{1, \bar{N}}$ are the Lagrange multiplayers, $g_{\bar{k}}$ are left hand side functions of the independent system of equations $\bar{M}$. Therefore the approximate solution of the original problem is the spline functions which minimizes the functional J (3.14). Notice that in the system of equations, which will appear as a consequence of minimizing (3.14), it is needed to use the equations $g_{\bar{k}} = 0$ too, according to the theory of Lagrange multiplayers.

Now let us consider the spline functions density for the one dimensional problem in detail, and briefly for the two dimensional problem, because of the similarity between the two problems.

The one-dimensional problem is reduced to find the function u(r) which must satisfy the differential equation (1):

(3.15)

$$Au = -\frac{d}{dr}\left(\frac{1}{r^3}\frac{du}{dr}\right) = 0,$$

with the boundary conditions $u(0) = 0, u(R) = T$.

The generalized solution of (3.15) is approximated by the following spline function

(3.16)
$$S_\Delta(r) = S_j(r) = a_j(r^4 - r_j^4) + d_j,$$
$$r_j < r < r_{j+1}, d_0 = 0.$$

$S_\Delta(r)$ is supposed to be continuous in $(0, R)$ and so from the connections between the polynomials (3.16), we get the following system of equations

(3.17)
$$a_j(r_{j+1}^4 - r_j^4) + d_j = d_{j+1}.$$

It can be seen that the properties and the definitions given for the spline functions in the two-dimensional problem can be applied for the one dimensional problem too. Therefore, we can get the approximate solution of (3.17) by using a similar functional to that of (3.14). The functions $g_{\bar{k}}$ in (3.14) now have the following forms

(3.18)
$$g_{\bar{k}} = a_{\bar{k}}(r_{\bar{k}+1}^4 - r_{\bar{k}}^4) + d_{\bar{k}} - d_{\bar{k}+1}.$$

Note it is easy to see that $S_\Delta(r) \in W_2^1 \cap C$ and $S_\Delta(r) \in H_A \cap C$.

THEOREM 2. The set of the splines $S_\Delta(r)$ (3.16) in a limit sense is dense in H_A, that is for every $f(r) \in H_A$ there exists $S_\Delta(r)$ such that

$$[f(r) - S_\Delta(r)] \to 0 \quad if \quad h_r \to 0(m \to \infty),$$

at the same time for $f(r)$ and $S_\Delta(r)$ the following are true

$$\| f(r) - S_\Delta(r)\|_{L_2} \to 0 \quad if \quad h_r \to 0,$$
$$\| f(r) - S_\Delta(r)\|_{W_2^1} \to 0 \quad if \quad h_r \to 0,$$

where $h_r = max \mid r_{j+1} - r_j \mid$.

PROOF. We remark that from Lemma 1 (which will be proved later on) it follows that $f(r) \in W_2^1$. Therefore there exists at least one point r_c in the interval (r_j, r_{j+1}) such that $\frac{df}{dr}(r_c)$ exists and bounded.

Let the form of the spline functions in (r_j, r_{j+1}):

(3.19)

$$S_\Delta(r) = S_j(r) := f_j + \frac{f_{j+1} - f_j}{r_{j+1}^4 - r_j^4}(r^4 - r_j^4),$$

where $f_j = f(r_j)$. It is obvious that (3.19) is continuous in all the interval $(0, R)$.

The Taylor's expansion in the neighborhood of $r_c in(r_j, r_{j+1})$ (6):

(3.20)

$$f(r) = f(r_c) + (r - r_c)\frac{df}{dr}(r_c) + o \mid r - r_c \mid,$$

where it can be written that $o \mid r - r_c \mid = 0 \mid r - r_c \mid^{d+1}, \quad 0 < d \leq 1$.

Formula (3.20) can be used to get the expansions of f_j, f_{j+1} in (3.19). Subtracting (3.19) from (3.20) we get

(3.21)

$$\mid f(r) - S_\Delta(r) \mid = \frac{df}{dr}(r_c) \cdot 0(h_r^2) + 0(h_r^{d+1}).$$

For the all interval $(0, R)$, we get

(3.22)

$$\| f(r) - S_\Delta(r) \|_{L_2} = 0(h_r^{d+1}).$$

The first derivatives of (3.19) and (3.20):

(3.23)

$$\frac{dS_j(r)}{dr} = \frac{f_{j+1} - f_j}{r_{j+1}^4 - r_j^4} 4r^3.$$

(3.24)

$$\frac{df(r)}{dr} = \frac{df}{dr}(r_c) + 0(| r - r_c |^d).$$

Subtracting (3.23) from (3.24), we obtain

(3.25)

$$| \frac{df(r)}{dr} - \frac{dS_j(r)}{dr} | = \frac{df}{dr}(r_c) \cdot 0(h) + 0(h^d),$$

and since $\frac{dS_\Delta(r)}{dr}$ are bounded in $(0, R)$ and $f(r) \in W_2^1$, we have

(3.26)

$$\| \frac{df(r)}{dr} - \frac{dS_\Delta(r)}{dr} \|_{L_2}(0, R) = 0(h^d).$$

Therefore from (3.22) and (3.26), we get

(3.27)

$$\| f(r) - S_\Delta(r) \|_{W_2^1} = 0(h^d).$$

For the norm in the H_A space, we have

(3.28)

$$[f(r) - S_\Delta(r)]^2 =:= \int_0^R \frac{1}{r^3}[\frac{d}{dr}(f(r) - S_\Delta(r))]^2 dr.$$

The norm (3.28) can be written as a sum of integrals over the subdomains (r_j, r_{j+1}). More precisely, because $f(r) \in H_A$, the Taylor's expansion of f in the neighborhood of $r = 0$:

(3.29)

$$f(r) = b_3 r^3 + b_4 r^4 + \dots,$$

and so from (3.23), (3.28) and (3.29), we get

(3.30)

$$\int_0^{h_1} \frac{1}{r^3}[\frac{d}{dr}(f(r) - S_0(r)]^2 dr = 0(h_r^2).$$

For $j > 0$ the integrals exists and bounded

(3.31)
$$\int_{r_j}^{r_{j+1}} \frac{1}{r^3}[\frac{d}{dr}(f(r) - S_\Delta(r)]^2 dr \leq$$

$$\leq \frac{1}{r_j^3} \int_{r_j}^{r_{j+1}} 0(h_r^{2d}) dr = 0(h_r^{2d+1}).$$

Finally the norm for all the interval can be written:

(3.32)
$$[f(r) - S_\Delta(r)]]^2 = \int_0^{h_1} \frac{1}{r^3}[\frac{d}{dr}(f(r) - S_0(r))]^2 dr +$$

$$+ \sum_{j=1}^{m-1} \int_{r_j}^{r_{j+1}} \frac{1}{r^3}[\frac{d}{dr}(f(r) - S_\Delta(r))]^2 dr =$$

$$= 0(h_r^2) + \sum_{j=1}^{m-1} h_r \cdot 0(h_r^{2d}) = 0(h_r^{2d}).$$

Therefore from (3.10) and (3.32) we get

(3.33)
$$[f(r) - S_\Delta(r)] = 0(h_r^d).$$

The proof is complete.

If $f(r) \in W_2^2 \cap H_A$, we can get a better estimations than that obtained in Theorem 2, and to show this we introduce the following theorem:

THEOREM 3. If $f(r) \in W_2^2 \cap H_A$, there exists $S_\Delta(r)$ such that the following estimations are true

$$[f(r) - S_\Delta(r)] = 0(h_r),$$
$$\| f(r) - S_\Delta(r)\|_{L_2} = 0(h_r^2),$$
$$\| f(r) - S_\Delta(r)\|_{W_2^1} = 0(h_r).$$

PROOF. Because $f(r) \in W_2^2$ there exists at least one point r_c on the interval (r_j, r_{j+1}) such that $\frac{d^2 f}{dr^2}(r_c)$ exists and bounded. The Taylor's expanstions of f at the neighborhood of r_c on (r_j, r_{j+1}):

(3.34)
$$f(r) = f(r_c) + \lambda_r h_r \frac{df}{dr}(r_c) + Q_1(r),$$

where

$$\lambda_r h_r = r - r_c, \quad | \lambda_r | \leq 1.$$

The Peano's form of the remainder in (3.34)[6,7]

$$Q_1(r) = \lambda_r^2 h_r^2 \frac{d^2 f}{dr^2}(r_c) + 0(h_r^2),$$

hence, we have that $| Q_1(r) | = 0(h_r^2)$.

Using for the spline functions the form (3.19) and subtract it from (3.34), we get

(3.35)
$$| f(r) - S_\Delta(r) | = 0(h_r^2).$$

For the all interval $(0, R)$, we get

(3.36)
$$\| f(r) - S_\Delta(r)\|_{L_2} = 0(h_r^2).$$

Subtracting $\frac{dS_\Delta(r)}{dr}$ from $\frac{df(r)}{dr}$, where the Peano's form for the remainder in expansion of $\frac{df}{dr}$ is $| Q_2(r) | = 0(h_r)$, we get

(3.37)

$$\left| \frac{df(r)}{dr} - \frac{dS_\Delta(r)}{dr} \right| = 0(h_r).$$

Since $\frac{dS_\Delta(r)}{dr}$ are bounded in $(0, R)$ and $f \in W_2^2$, we obtain

(3.38)

$$\left\| \frac{df(r)}{dr} - \frac{dS_\Delta(r)}{dr} \right\|_{L_2(0,R)} = 0(h_r).$$

Therefore from (3.36) and (3.38), we get

(3.39)

$$\| f(r) - S_\Delta(r) \|_{W_2^1} = 0(h_r).$$

Following Theorem 2, we can get the norm in H_A space:

(3.40)

$$[f(r) - S_\Delta(r)]^2 = \int_0^{h_1} \frac{1}{r^3} [\frac{d}{dr}(f(r) - S_0(r))]^2 dr +$$

$$+ \sum_{j=1}^{m-1} \int_{r_j}^{r_{j+1}} \frac{1}{r^3} [\frac{d}{dr}(f(r) - S_\Delta(r))]^2 dr =$$

$$= 0(h_r^2) + \sum_{j=1}^{m-1} h_r \cdot 0(h_r^2) = 0(h_r^2).$$

Therefore from (3.10) and (3.40), we get

(3.41)

$$[f(r) - S_\Delta(r)] = 0(h_r).$$

The proof is complete.

The density of the spline functions for the two-dimensional problem will be introduced according to the following Theorem:

THEOREM 4. The set of the spline functions $S_\Delta(z,r)$ (2.5) in a limit sense is dense in H_A space. More over, if $f(z,r) \in W_2^2 \cap H_A$, the following estimations are true

$$[f(z,r) - S_\Delta(z,r)] = 0(h_z + h_r),$$
$$\| f(z,r) - S_\Delta(z,r)\|_{L_2} = 0(h_z^2 + h_r^2),$$
$$\| f(z,r) - S_\Delta(z,r)\|_{W_2^1} = 0(h_z + h_r),$$

where $h_z = max \mid z_{i+1} - z_i \mid, h_r = max \mid r_{j+1} - r_j \mid$.

The proof of this theorem can be accomplished in a similar manner used in the proofs of Theorems 2 and 3.

4. Convergence of the approximate solutions.

LEMMA 1. In the one-dimensional problem (3.16), for every $u(r) \in H_A$ there exists a constant c_1 such that

$$\| u\|_{W_2^1} \leq c_1[u].$$

PROOF. For the L_2-norm of the first derivative of the function u on the interval $(0,R)$, we get

(4.1)
$$\| \frac{du}{dr}\|_{L_2}^2 = \int_0^R \mid \frac{du}{dr} \mid^2 dr \leq$$
$$\leq R^3 \int_0^R \frac{1}{r^3} \mid \frac{du}{dr} \mid^2 = R^3[u,u].$$

Let

(4.2)
$$u = \int_0^r \frac{du}{dx} dx = \int_0^r x^{3/2}[\frac{1}{x^{3/2}} \frac{du}{dx}]dx.$$

Using Cauchy-Bunyakovsky's inequality to the integral (4.2)
(4.3)

$$u^2 \leq \int_0^r x^3 dx \int_0^R (\frac{1}{x^{3/2}} \mid \frac{du}{dx} \mid)^2 dx \leq \frac{r^4}{4} \int_0^R \frac{1}{x^3} \mid \frac{du}{dx} \mid^2 dx,$$

we get
(4.4)

$$\| u\|_{L_2}^2 = \int_0^R u^2 dr \leq \int_0^R \frac{r^4}{4} dr[u,u] = \frac{R^5}{20}[u]^2.$$

Therefore, from (4.1) and (4.4), we obtain the norm of the function u in W_2^1.
(4.5)

$$\| u\|_{W_2^1}^2 \leq (R^3 + \frac{R^5}{20})[u]^2.$$

Let
(4.6)

$$c_1 = R\sqrt{R + \frac{R^3}{20}},$$

then, from (4.5) and (4.6), we get
(4.7)

$$\| u\|_{W_2^1} \leq c_1[u].$$

The proof is complete.

LEMMA 2. In the two-dimensional problem (1.1), for every $u(z,r) \in H_A$ there exists a constant c_2 such that

$$\| u\|_{W_2^1} \leq c_2[u].$$

PROOF. For the L_2-norm of the first derivative of the function $u(z,r)$ in Ω, we get

(4.8)
$$\| \frac{\partial u}{\partial z} \|_{L_2}^2 \leq R^3 \int_\Omega \frac{1}{r^3} \left| \frac{\partial u}{\partial z} \right|^2 dzdr.$$

(4.9)
$$\| \frac{\partial u}{\partial r} \|_{L_2}^2 \leq R^3 \int_\Omega \frac{1}{r^3} \left| \frac{\partial u}{\partial r} \right|^2 dzdr.$$

Therefore, from (4.8) and (4.9), we obtain

(4.10)
$$\| \frac{\partial u}{\partial z} \|_{L_2}^2 + \| \frac{\partial u}{\partial r} \|_{L_2}^2 \leq R^3 [u, u].$$

Let

(4.11)
$$u(z,r) = \int_0^r \frac{\partial u}{\partial x}(z,x)dx = \int_0^r x^{3/2} \frac{1}{x^{3/2}} \frac{\partial u}{\partial x}(z,x)dx.$$

Using Cauchy-Bunyakovsky's inequality to the integral (4.11)

(4.12)
$$u^2(z,r) \leq \int_0^r x^3 dx \int_0^r \frac{1}{x^3} \left| \frac{\partial u}{\partial x}(z,x) \right|^2 dx \leq$$
$$\leq \frac{r^4}{4} \int_0^R \frac{1}{x^3} \left| \frac{\partial u}{\partial x}(z,x) \right|^2 dx.$$

Integrating the both sides of (4.12) over the domain Ω, we get

(4.13)
$$\int_\Omega u^2(z,r)dzdr \leq \int_0^R \frac{r^4}{4} dr \cdot \int_0^R \int_0^Z \frac{1}{x^3} \left| \frac{\partial u}{\partial x} \right|^2 dzdx =$$

$$= \frac{R^5}{20} \int_{\Omega} \frac{1}{r^3} \mid \frac{\partial u}{\partial r} \mid^2 dzdr.$$

Let

(4.14)

$$u(z,r) = \int_0^z \frac{\partial u}{\partial y}(z,r)dy = r^{3/2} \int_0^z 1 \cdot \frac{1}{r^{3/2}} \mid \frac{\partial u}{\partial y}(y,r) \mid dy.$$

Using Cauchy-Bunyakovsky's inequality to the integral

(4.15)

$$u^2(z,r) \le r^3 z \int_0^z \frac{1}{r^3} \mid \frac{\partial u}{\partial y} \mid^2 dy \le R^3 z \int_0^Z \frac{1}{r^3} \mid \frac{\partial u}{\partial y} \mid^2 dy.$$

Integrating the both sides of (4.15) over the domain Ω, we get

(4.16)

$$\int_{\Omega} u^2(z,r)dzdr \le \frac{R^3 Z^2}{2} \int_{\Omega} \frac{1}{r^3} \mid \frac{\partial u}{\partial z} \mid^2 dzdr.$$

Therefore, from (4.13) and (4.16), we obtain

(4.17)

$$2 \int_{\Omega} u^2(z,r)dzdr \le max(\frac{R^5}{20}, \frac{R^3 Z^2}{2})[u,u].$$

Hence

(4.18)

$$\| u \|^2 \le max(\frac{R^5}{40}, \frac{R^3 Z^2}{4})[u,u].$$

Finally, from (4.10) and (4.18), we get the norm of the function u in w_2^1

(4.19)

$$\| u\|^2_{W^1_2} \le R^3(1 + max(\frac{R^2}{40}, \frac{Z^2}{4}))[u, u].$$

The proof is complete.

THEOREM 5. The approximated solutions $\tilde{S}$, obtained by the minimization of the functional (3.14) over a family of splines, individualy, for the one and two dimensional problems are convergent to the generalized solutions in the W_2^1 norm:

$$\| \tilde{u}_0 - \tilde{S} \|_{W_2^1} \to 0, \quad if \quad h \to 0.$$

where $\tilde{u}_0$ are denote the generalized solutions of the original differential problems (1.1) and (3.15), and let $h = max \mid h_r \mid$ or $h = max \mid h_z, h_r \mid$.

PROOF. The generalized solutions $\tilde{u}_0$ satisfy the following equality

(4.20)

$$[\tilde{u}_0 - \tilde{S}]^2 = [\tilde{u}_0 - \tilde{S}, \tilde{u}_0 - \tilde{S}] = J(\tilde{S}) - J(\tilde{u}_0).$$

Because $\tilde{S}$ are minimizing the functional J, for every spline function S_Δ, we have

(4.21)

$$J(\tilde{S}) - J(\tilde{u}_0) \le J(S_\Delta) - J(\tilde{u}_0) = [\tilde{u}_0 - S_\Delta]^2,$$

therefore, from (4.20) and (4.21), we obtain

(4.22)

$$[\tilde{u}_0 - \tilde{S}] \le \inf_{S_\Delta}[\tilde{u}_0 - S_\Delta].$$

Hence, from (4.22), Lemma 1 and Theorem 2, respectively, from (4.22), Lemma 2 and Theorem 4, we have

(4.23)

$$\| \tilde{u}_0 - \tilde{S}\|_{W_2^1} \leq C[\tilde{u}_0 - \tilde{S}] \leq C \inf_{S_\Delta}[\tilde{u}_0 - S_\Delta] \to 0$$

$$if \quad h \to 0.$$

The proof is complete.

Remark 3. We denote that if $\tilde{u}_0$ are the classical solutions of the problems (1.1)-(1.5) and (3.15), it mean, $\tilde{u}_0 \in D(A)$ and at the same time $\tilde{u}_0 \in W_2^2$. Therefore, in this case, Theorems 3 and 4 can be used with (4.22), Lemma 1, Lemma 2 to obtain the better estimations for $\| \tilde{u}_0 - \tilde{S} \|$:

$$\| \tilde{u}_0 - \tilde{S}\|_{W_2^1} = 0(h).$$

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