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ON STURM-LIOUVILLE DIFFERENCE EQUATIONS*

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Many papers are devoted to the study of the Sturm-Liouville equation by the corresponding difference equations, see [1-6]. In [1-3] the authors investigate the convergence and the speed of convergence in the class of smooth coefficients for special systems of nodes. In [4-5] homogeneous difference schemes are used for the solutions of Sturm-Liouville equations in the class $Q^{(m)}$ of singular /i.e. not continuous/ coefficients.

In the present paper we investigate the approximation of Sturm-Liouville equations by difference equations /local error and convergence/. We shall use some result of V.A. Il'in and I. Joó [9-11].

1. Local error of approximation

1.1. Local error for the equation

$$-\mathbf{u}'' + \mathbf{q}(\mathbf{x})\mathbf{u} = \lambda\mathbf{u} \qquad \lambda \geq 0 \tag{1}$$

in case of equidistant nodes.

Consider the Sturm-Liouville problem (1) on an interval G /finite or not/. Suppose that

$$q \in L^{\infty}(G).$$

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*Some results of this paper are published in [15] without proofs.

As a solution we consider $u \in L^2(G)$ such that u and u' are absolutely continuous on G and (1) holds a.e. on G. Let $h > 0$ be the distance of nodes and approximate the equation (1) by the difference equation

$$-y_{\bar{x}x} + d(x)y = \lambda y \tag{2}$$

where

$$y_{\bar{x}x} := [y(x+h) + y(x-h) - 2y(x)]/h^2$$

$$d(x) := \frac{1}{h^2} \int_0^h [q(x+t) + q(x-t)](h-t)dt. \tag{3}$$

Here and in what follows we use the notation of the books [7-8]. Let

$$Lu := -u'' + qu - \lambda u$$

and the corresponding difference equation

$$\Lambda u := -y_{\bar{x}x} + dy - \lambda y.$$

Theorem 1 [12]

If the above conditions hold, then the difference operator Λ approximates the operator L on the first degree for the solutions of (1). In other words,

$$Lu = 0 \Rightarrow \Lambda u = \underline{\underline{O}}(h).$$

If moreover $q \in Lip1$, then

$$Lu = 0 \Rightarrow \Lambda u = \underline{\underline{O}}(h^2).$$

For the proof we need

Lemma 1 ([9],[11])

Let G be a finite interval and $q \in L^1(G)$, then (for complex eigenvalues λ)

$$\| u \|_{L^\infty(G)} \leq c(1+ \mid Im\sqrt{\lambda} \mid)^{\frac{1}{2}} \| u \|_{L^2(G)}, \tag{4}$$

$$\| u' \|_{L^\infty(G)} \leq c(1+ \mid \sqrt{\lambda} \mid) \| u \|_{L^\infty(G)}. \tag{5}$$

We shall also use the Titchmarsh formula

$$\frac{u(x+h)+u(x-h)}{2} = u(x)cos\sqrt{\lambda}h - \frac{1}{2\sqrt{\lambda}} \int_{x-h}^{x+h} q(\xi)u(\xi)sin\sqrt{\lambda}(\mid x-\xi \mid -h)d\xi \tag{6}$$

if $x \pm h \in G$ and $\lambda \neq 0$: for $\lambda = 0$ the limit equality

$$\frac{u(x+h)+u(x-h)}{2} = u(x) - \frac{1}{2} \int_{x-h}^{x+h} q(\xi)u(\xi)(\mid x-\xi \mid -h)d\xi \tag{7}$$

holds. We shall deal only with the case $\lambda \neq 0$.

By the Titchmarsh formula (6)

$$u_{\bar{x}x} = [u(x+h)+u(x-h)-2u(x)]/h^2 =$$

$$= 2u(x)\frac{cos\sqrt{\lambda}h - 1}{h^2} - \int_{x-h}^{x+h} q(\xi)u(\xi)\frac{sin\sqrt{\lambda}(|x - \xi| - h)}{\sqrt{\lambda}h^2}d\xi$$

hence

$$-\Lambda u = u_{xx} - du + \lambda u = 2u(x)\frac{cos\sqrt{\lambda}h - 1 + \lambda h^2/2}{h^2} - \tag{8}$$

$$-\int_0^h [q(x+t)u(x+t)+q(x-t)u(x-t)]\frac{sin\sqrt{\lambda}(t-h) - \sqrt{\lambda}(t-h)}{\sqrt{\lambda}h^2}dt$$

$$-\int_0^h [q(x+t)(u(x+t) - u(x)) - q(x-t)(u(x) - u(x-t))]\frac{t-h}{h^2}dt$$

$$=: I_1 + I_2 + I_3.$$

It is easy to see that

$$| I_1 | \leq c\lambda^2 h^2 \, | u(x) | \tag{9}$$

$$| I_2 | \leq \frac{c}{\sqrt{\lambda}h^2} \int_0^h (\sqrt{\lambda}t)^3 dt \cdot \| u \|_{L^\infty(x-h,x+h)} \leq$$

$$\leq c\lambda h^2 \| u \|_{L^\infty(x-h,x+h)} . \tag{10}$$

Using the fact that

$$u(x \pm t) - u(x) = tu'(\xi_\pm) \quad \text{for some} \begin{cases} x < \xi_+ < x+t \\ x-t < \xi_- < x \end{cases}$$

we obtain by (5) that

$$| I_3 | \leq \frac{c}{\sqrt{\lambda}h^2} \int_0^h \sqrt{\lambda}ht \| u' \|_{L^\infty(x-t,x+t)} dt \leq$$

$$\leq c(1+\sqrt{\lambda})h \| u \|_{L^\infty(x-h,x+h)} \tag{11}$$

so the first statement of Theorem 1 is proved.

To show the second one, rewrite I_3 in the form

$$I_3 = \frac{-1}{\sqrt{\lambda}h^2} \int_0^h \sqrt{\lambda}(t-h)t[q(x+t)u'(\xi_+) - q(x-t)u'(\xi_-)]dt =$$

$$= -\int_0^h \frac{(t-h)t}{h^2} u'(\xi_+)[q(x+t) - q(x)]dt$$

$$-\int_0^h \frac{(t-h)t}{h^2} u'(\xi_-)[q(x)-q(x-t)]dt$$

$$-\int_0^h \frac{(t-h)t}{h^2} q(x)[u'(\xi_+)-u'(\xi_-)]dt.$$

By (1) we have

$$|u'(\xi_+)-u'(\xi_-)| \le h\, ||u''||_{L^\infty(x-h,x+h)} \le$$

$$\le ch(1+\lambda)\, ||u||_{L^\infty(x-h,x+h)}$$

and hence

$$|I_3| \le c\int_0^h h(1+\lambda)dt\cdot ||u||_{L^\infty(x-h,x+h)} \le$$

$$\le c(1+\lambda)h^2\, ||u||_{L^\infty(x-h,x+h)}\,. \tag{12}$$

Theorem 1 is completely proved.

1.2 Local error estimate for the equation $-u'' + qu = \lambda u$ with non-equidistant nodes.

In this point we approximate (1) by the difference equation

$$-y_{x\hat{x}} + dy = \lambda y \tag{13}$$

where

$$y_{x\hat{x}} := \frac{2}{h_+ + h_-}\left[\frac{y(x+h_+) - y(x)}{h_+} - \frac{y(x) - y(x-h_-)}{h_-}\right],$$

$$d(x) := \frac{2}{h_+ + h_-}\left[\frac{1}{h_+}\int_0^{h_+} q(x+t)(h_+ - t)dt + \right.$$

$$\tag{14}$$

$$\left. + \frac{1}{h_-}\int_0^{h_-} q(x-t)(h_- - t)dt\right],$$

$$x + h_+, x - h_- \in G.$$

We have

Theorem 2

The difference operator $\Lambda := -y_{x\hat{x}} + dy - \lambda y$ approximates L on the first degree for the solutions of (1),

$$Lu = 0 \Rightarrow \Lambda u = \underline{\underline{0}}(\bar{h}), \quad \bar{h} := max\{h_+, h_-\}.$$

Proof

We need the asymmetric Titchmarsh formulae

$$u(x+h_+) = u(x)cos\sqrt{\lambda}h_+ + u'(x)\frac{sin\sqrt{\lambda}h_+}{\sqrt{\lambda}}$$

$$\tag{15}$$

$$-\int_{x}^{x+h_+} q(\xi)u(\xi)\frac{sin\sqrt{\lambda}(\xi - x - h_+)}{\sqrt{\lambda}}d\xi,$$

$$u(x-h_-) = u(x)cos\sqrt{\lambda}h_- - u'(x)\frac{sin\sqrt{\lambda}h_-}{\sqrt{\lambda}} -$$

$$-\int_{x-h_-}^{x} q(\xi)u(\xi)\frac{sin\sqrt{\lambda}(x - \xi - h_-)}{\sqrt{\lambda}}d\xi.$$

Hence

$$\frac{u(x+h_+)-u(x)}{h_+} = u(x)\frac{cos\sqrt{\lambda}h_+ - 1}{h_+} + u'(x)\frac{sin\sqrt{\lambda}h_+}{\sqrt{\lambda}h_+} -$$

$$-\int_0^{h_+} q(x+t)u(x+t)\frac{sin\sqrt{\lambda}(t-h_+)}{\sqrt{\lambda}h_+}dt,$$

$$\frac{u(x-h_-)-u(x)}{h_-} = u(x)\frac{cos\sqrt{\lambda}h_- - 1}{h_-} - u'(x)\frac{sin\sqrt{\lambda}h_-}{\sqrt{\lambda}h_-} -$$

$$-\int_0^{h_-} q(x-t)u(x-t)\frac{sin\sqrt{\lambda}(t-h_-)}{\sqrt{\lambda}h_-}dt$$

and then

$$u_{x\hat{x}} - du + \lambda u = \frac{2}{h_+ + h_-}\left\{u(x)\left[\frac{cos\sqrt{\lambda}h_+ - 1 + \lambda h_+^2/2}{h_+}\right.\right.$$

$$+\frac{\cos\sqrt{\lambda}h_- - 1 + \lambda h_-^2/2}{h_-}\Bigg] + u'(x)\left[\frac{\sin\sqrt{\lambda}h_+}{\sqrt{\lambda}h_+} - \frac{\sin\sqrt{\lambda}h_-}{\sqrt{\lambda}h_-}\right] -$$

$$-\int_0^{h_+} q(x+t)u(x+t)\frac{\sin\sqrt{\lambda}(t-h_+) - \sqrt{\lambda}(t-h_+)}{\sqrt{\lambda}h_+}dt -$$

$$-\int_0^{h_+} q(x+t)[u(x+t)-u(x)]\frac{\sqrt{\lambda}(t-h_+)}{\sqrt{\lambda}h_+}dt -$$

$$-\int_0^{h_-} q(x-t)u(x-t)\frac{\sin\sqrt{\lambda}(t-h_-) - \sqrt{\lambda}(t-h_-)}{\sqrt{\lambda}h_-}dt -$$

$$-\int_0^{h_-} q(x-t)[u(x-t)-u(x)]\frac{\sqrt{\lambda}(t-h_-)}{\sqrt{\lambda}h_-}dt\Bigg\} =$$

$$=: \frac{2}{h_+ + h_-}\{I_1 + I_2 + I_3 + I_4 + \hat{I}_3 + \hat{I}_4\}.$$

We know that

$$| I_1 | \le c\lambda^2 \bar{h}^3 \, | u(x) |,$$

$$| I_2 | \le c\lambda^{\frac{3}{2}} \bar{h}^2 \frac{| u'(x) |}{\sqrt{\lambda}},$$

$$| I_3 | \leq \frac{c}{\sqrt{\lambda} h_+} \int_0^{h_+} [\sqrt{\lambda}(h_+ - t)]^3 dt \cdot \| u \|_{L^\infty(x,x+h_+)} \leq$$

$$\leq c\lambda \bar{h}^3 \| u \|_{L^\infty(x,x+h_+)},$$

$$| \hat{I}_3 | \leq c\lambda \bar{h}^3 \| u \|_{L^\infty(x-h_-,x)},$$

$$| I_4 | \leq \frac{c}{\sqrt{\lambda} h_+} \int_0^{h_+} | u'(\xi_+) | t\sqrt{\lambda}(h_+ - t) dt \leq$$

$$\leq c(1+\sqrt{\lambda})\bar{h}^2 \| u \|_{L^\infty(x,x+h_+)},$$

$$| \hat{I}_4 | \leq c(1+\sqrt{\lambda})\bar{h}^2 \| u \|_{L^\infty(x-h_-,x)} .$$

The above estimates prove Theorem 2.

1.3 Local error of the approximation of the equation

$$-(\mathbf{p}(\mathbf{x})\mathbf{u}')' + \mathbf{q}(\mathbf{x})\mathbf{u} = \lambda \mathbf{u}.$$

We consider the eigenfunctions of the above selfadjoint Sturm-Liouville operator with singular coefficient on a finite interval (a,b). Let $x_0 \in (a,b)$ be fixed and suppose that

$$0 < \alpha \leq p_1(x) \in C^2(a, x_0], \quad 0 < \beta \leq p_2(x) \in C^2[x_0, b)$$

$$q_1(x) \in C(a, x_0], \quad q_2(x) \in C[x_0, b)$$

and

$$p(x) := \begin{cases} p_1(x) & x < x_0 \\ p_2(x) & x > x_0, \end{cases}$$

$$q(x) := \begin{cases} q_1(x) & x < x_0 \\ q_2(x) & x > x_0. \end{cases}$$

A function $u \not\equiv 0$ is an eigenfuction with the eigenvalue $\lambda \geq 0$ if

$$u(x) \in C[a,b], C^1[a,x_0], C^1[x_0,b], C^2(a,x_0), C^2(x_0,b)$$

further

$$\begin{aligned} -(p_1 u')' + q_1 u = \lambda u \qquad \text{if} \quad a < x < x_0 \\ -(p_2 u')' + q_2 u = \lambda u \qquad \text{if} \quad x_0 < x < b \end{aligned} \tag{16}$$

finally if the derivatives satisfy

$$p_1(x) u'(x_0 - 0) = p_2(x) u'(x_0 + 0). \tag{17}$$

We apply the following non-equidistant difference scheme

$$u_{x\hat{x}} := \frac{2}{\sqrt{p_2(x_0)}h_+ + \sqrt{p_1(x_0)}h_-}$$

$$\left[\frac{\sqrt{p_2(x_0 + \varrho_2(h_+))}(u(x_0 + \varrho_2(h_+)) - u(x_0))}{h_+} \right.$$

$$\left. + \frac{\sqrt{p_1(x_0 - \varrho_1(h_-))}(u(x_0 - \varrho_1(h_-)) - u(x_o))}{h_-} \right],$$

where the functions $\varrho_1(t), \varrho_2(t)$ are defined for $t > 0$ by

$$\int_{x_0-\varrho_1(t)}^{x_0} \frac{1}{\sqrt{p_1}} = t, \qquad \int_{x_0}^{x_0+\varrho_2(t)} \frac{1}{\sqrt{p_2}} = t. \tag{18}$$

The Titchmarsh type formula we shall use here is essentially given in [10]. Namely consider the function

$$w(x) := \begin{cases} \frac{1}{\sqrt{\lambda}} sin\sqrt{\lambda}\left(\int_{x_0}^{x} \frac{1}{\sqrt{p_2}} - h_+\right) & x_0 < x \leq x_0 + \varrho_2(h_+), \\ \frac{1}{\sqrt{\lambda}} sin\sqrt{\lambda}\left(\int_{x}^{x_0} \frac{1}{\sqrt{p_1}} - h_-\right) & x_0 - \varrho_1(h_-) \leq x < x_0. \end{cases}$$

Then we have

$$(pw')' = -\lambda w + \begin{cases} \frac{p_2'(x)}{2\sqrt{p_2(x)}} cos\sqrt{\lambda}\left(\int_{x_0}^{x} \frac{1}{\sqrt{p_2}} - h_+\right) \\ -\frac{p_1'(x)}{2\sqrt{p_1(x)}} cos\sqrt{\lambda}\left(\int_{x}^{x_0} \frac{1}{\sqrt{p_1}} - h_-\right). \end{cases} \tag{19}$$

Twofold integration by parts gives

$$\int_{x_0}^{x_0+\varrho_2(h_+)} [(pw')'u - (pu')'w] = [p_2 w'u - p_2 wu']_{x_0}^{x_0+\varrho_2(h_+)} =$$

$$= \sqrt{p_2(x_0 + \varrho_2(h_+))}u(x_0 + \varrho_2(h_+)) - \sqrt{p_2(x_0)}u(x_0)cos\sqrt{\lambda}h_+ -$$

$$-p_2(x_0)u'(x_0 + 0)\frac{sin\sqrt{\lambda}h_+}{\sqrt{\lambda}},$$

$$\int\limits_{x_0-\varrho_1(h_-)}^{x_0} [(pw')'u-(pu')'w] = [p_1w'u-p_1wu']_{x_0-\varrho_1(h_-)}^{x_0} =$$

$$= \sqrt{p_1(x_0-\varrho_1(h_-))}u(x_0-\varrho_1(h_-)) - \sqrt{p_1(x_0)}u(x_0)cos\sqrt{\lambda}h_- +$$

$$+p_1(x_0)u'(x_0-0)\frac{sin\sqrt{\lambda}h_-}{\sqrt{\lambda}}.$$

Taking into account (19), (17) and the equality $-\lambda u-(pu')' = -qu$ we obtain that

$$u_{x\hat{x}} = \frac{2}{\sqrt{p_2(x_0)}h_+ + \sqrt{p_1(x_0)}h_-}. \tag{20}$$

$$\cdot\Big\{p_2(x_0)u'(x_0+0)\big[\frac{sin\sqrt{\lambda}h_+}{\sqrt{\lambda}h_+} - \frac{sin\sqrt{\lambda}h_-}{\sqrt{\lambda}h_-}\big] + \sqrt{p_2(x_0)}u(x_0).$$

$$\cdot\frac{cos\sqrt{\lambda}h_+ - 1 - \lambda h_+^2/2}{h_+} + \sqrt{p_1(x_0)}u(x_0)\frac{cos\sqrt{\lambda}h_- - 1 - \lambda h_-^2/2}{h_-}$$

$$-\lambda u(x_0)\big[\sqrt{p_2(x_0)}\frac{h_+}{2} + \sqrt{p_1(x_0)}\frac{h_-}{2}\big]$$

$$-\int\limits_{x_0}^{x_0+\varrho_2(h_+)} q_2(x)u(x)\frac{sin\sqrt{\lambda}\big(\int\limits_{x_0}^{x}\frac{1}{\sqrt{p_2}} - h_+\big)}{\sqrt{\lambda}h_+}dx$$

$$-\int\limits_{x_0-\varrho_1(h_-)}^{x_0} q_1(x)u(x)\frac{sin\sqrt{\lambda}\big(\int\limits_{x}^{x_0}\frac{1}{\sqrt{p_1}} - h_-\big)}{\sqrt{\lambda}h_-}dx$$

$$+\frac{1}{h_+}\int_{x_0}^{x_0+\varrho_2(h_+)}\left[u(x)cos\sqrt{\lambda}\left(\int_{x_0}^{x}\frac{1}{\sqrt{p_2}}-h_+\right)-u(x_0)\right]\frac{p_2'(x)}{2\sqrt{p_2(x)}}dx$$

$$-\frac{1}{h_-}\int_{x_0-\varrho_1(h_-)}^{x_0}\left[u(x)cos\sqrt{\lambda}\left(\int_{x}^{x_0}\frac{1}{\sqrt{p_1}}-h_-\right)-u(x_0)\right]\frac{p_2'(x)}{2\sqrt{p_2(x)}}dx.$$

Define now the difference analogon of the potential q by

$$d(x_0):=\frac{2}{\sqrt{p_2(x_0)}h_+ + \sqrt{p_1(x_0)}h_-}\left\{\frac{1}{h_+}\int_{x_0}^{x_0+\varrho_2(h_+)} q_2(x)\right.$$

$$\left(\int_{r_0}^{x}\frac{1}{\sqrt{p_2}}-h_+\right)dx+$$

$$\left.+\frac{1}{h_-}\int_{x_0-\varrho_1(h_-)}^{x_0} q_1(x)\left(\int_{x}^{x_0}\frac{1}{\sqrt{p_1}}-h_-\right)dx\right\}.$$

Theorem 3

Let

$$Lu := -(pu')' + qu - \lambda u$$

$$\Lambda u := -u_{x\hat{x}} + du - \lambda u.$$

Suppose further that $p_1'(x_0)p_2'(x_0) < 0$. Then in the discontinuity point x_0 we have an approximation of first degree:

$$Lu = 0 \Rightarrow \Lambda u(x_0) = \underline{\underline{O}}(\bar{h}), \quad \bar{h} := max\{h_+, h_-\},$$

whenever we chose h_+, h_- satisfying

$$h_+ \frac{p_2'(x_0)}{\sqrt{p_2(x_0)}} + h_- \frac{p_1'(x_0)}{\sqrt{p_1(x_0)}} = 0. \tag{21}$$

Proof

From (20) we get the equality

$$u_{x\hat{x}} - du + \lambda u = \frac{2}{\sqrt{p_2(x_0)}h_+ + \sqrt{p_1(x_0)}h_-}\cdot$$

$$\cdot\Bigg\{p_2(x_0)u'(x_0+0)\left[\frac{sin\sqrt{\lambda}h_+}{\sqrt{\lambda}h_+} - \frac{sin\sqrt{\lambda}h_-}{\sqrt{\lambda}h_-}\right]+$$

$$+u(x_0)\left[\sqrt{p_2(x_0)}\frac{cos\sqrt{\lambda}h_+ - 1 + \lambda h_+^2/2}{h_+}+\right.$$

$$\left.+\sqrt{p_1(x_0)}\frac{cos\sqrt{\lambda}h_- - 1 + \lambda h_-^2/2}{h_-}\right]-$$

$$-\int_0^{e_2(h_+)} q_2(x_0+t)u(x_0+t)$$

$$\frac{sin\sqrt{\lambda}\left(\int_{x_0}^{x_0+t}\frac{1}{\sqrt{p_2}} - h_+\right) - \sqrt{\lambda}\left(\int_{x_0}^{x_0+t}\frac{1}{\sqrt{p_2}} - h_+\right)}{\sqrt{\lambda}h_+}dt-$$

$$-\int_{0}^{\varrho_1(h_-)} q_1(x_0-t)u(x_0-t)$$

$$\frac{sin\sqrt{\lambda}(\int_{x_0-t}^{x_0}\frac{1}{\sqrt{p_1}}-h_-)-\sqrt{\lambda}(\int_{x_0-t}^{x_0}\frac{1}{\sqrt{p_1}}-h_-)}{\sqrt{\lambda}h_-}dt-$$

$$-\int_{0}^{\varrho_2(h_+)} q_2(x_0+t)[u(x_0+t)-u(x_0)]\frac{\int_{x_0}^{x_0+t}\frac{1}{\sqrt{p_2}}-h_+}{h_+}dt-$$

$$-\int_{0}^{\varrho_1(h_-)} q_1(x_0-t)[u(x_0-t)-u(x_0)]\frac{\int_{x_0-t}^{x_0}\frac{1}{\sqrt{p_1}}-h_-}{h_-}dt+$$

$$+\frac{1}{h_+}\int_{0}^{\varrho_2(h_+)}\frac{p_2'(x_0+t)}{2\sqrt{p_2(x_0+t)}}[u(x_0+t)cos\sqrt{\lambda}$$

$$(\int_{x_0}^{x_0+t}\frac{1}{\sqrt{p_2}}-h_+)-u(x_0)]dt-$$

$$-\int_{0}^{\varrho_1(h_-)}\frac{1}{h_-}\frac{p_1'(x_0-t)}{2\sqrt{p_1(x_0-t)}}$$

$$[u(x_0-t)cos\sqrt{\lambda}(\int_{x_0-t}^{x_0}\frac{1}{\sqrt{p_1}}-h_-)-u(x_0)]dt=$$

$$=: \frac{2}{\sqrt{p_2(x_0)}h_+ + \sqrt{p_1(x_0)}h_-}\{I_1 + I_2 + I_3 + \hat{I}_3 + I_4 + \hat{I}_4 + I_5 + \hat{I}_5\}.$$

From the expansions of *sinx* and *cosx* we see that with the notation

$$K := [x_0 - \varrho_1(h_-), x_0 + \varrho_2(h_+)]$$

we have

$$| I_1 | \le c(\lambda + 1)^{3/2}\bar{h}^2 \parallel u \parallel_{L^\infty(K)},$$

$$| I_2 | \le c\lambda^2 \bar{h}^3 \mid u(x_0) \mid .$$

Now if $\bar{h}$ is small enough then the segment K is not too close to the boundary points a and b, hence $\frac{1}{\sqrt{p_i}}, i = 1,2$ are bounded from below and then

$$h_+ = \int_{x_0}^{x_0+\varrho_2(h_+)} \frac{1}{\sqrt{p_2}} \ge c\varrho_2(h_+),$$

$$h_- = \int_{x_0-\varrho_1(h_-)}^{x_0} \frac{1}{\sqrt{p_1}} \ge c\varrho_1(h_-).$$

Using this we see that

$$| I_3 | \le c\lambda h_+^2 \varrho_2(h_+) \parallel u \parallel_{L^\infty(K)} \le$$

$$\le c\lambda \bar{h}^3 \parallel u \parallel_{L^\infty(K)},$$

$$| \hat{I}_3 | \leq c\lambda h_-^2 \varrho_1(h_-) \| u \|_{L^\infty(K)} \leq$$

$$\leq c\lambda \bar{h}^3 \| u \|_{L^\infty(K)} .$$

By (5) we get then

$$| I_4 |, | \hat{I}_4 | \leq c(1+\sqrt{\lambda})\bar{h}^2 \| u \|_{L^\infty(K)} .$$

To estimate I_5 we take a decomposition of the integrand:

$$\left| \int_0^{\varrho_2(h_+)} u(x_0+t) \frac{p_2'(x_0+t)}{2\sqrt{p_2(x_0+t)}} \frac{\cos\sqrt{\lambda}\left(\int_{x_0}^{x_0+t} \frac{1}{\sqrt{p_2}} - h_+\right) - 1}{h_+} dt \right| \leq$$

$$\leq c\lambda \bar{h}^2 \| u \|_{L^\infty(K)},$$

$$\left| \int_0^{\varrho_2(h_+)} [u(x_0+t) - u(x_0)] \left(\frac{p_2'(x_0+t)}{2\sqrt{p_2(x_0+t)}} - \frac{p_2'(x_0)}{2\sqrt{p_2(x_0)}} \right) \frac{1}{h_+} dt \right| \leq$$

$$\leq c(1+\lambda)\bar{h}^2 \| u \|_{L^\infty(K)}$$

which shows that

$$\left| I_5 - \frac{1}{h_+} \int_0^{\varrho_2(h_+)} [u(x_0+t) - u(x_0)] \frac{p_2'(x_0)}{2\sqrt{p_2(x_0)}} dt \right| \leq$$

$$\leq c(1+\lambda)\bar{h}^2 \parallel u \parallel_{L^\infty(K)} .$$

Analogously

$$| \hat{I}_5 + \frac{1}{h_-} \int_0^{\varrho_1(h_-)} [u(x_0 - t) - u(x_0)] \frac{p_1'(x_0)}{2\sqrt{p_1(x_0)}} dt | \leq$$

$$\leq c(1+\lambda)\bar{h}^2 \parallel u \parallel_{L^\infty(K)} .$$

We know that

$$| u(x_0 + t) - u(x_0) - tu'(x_0 + 0) | \leq ct^2 \parallel u'' \parallel_{L^\infty(K)} \leq$$

$$\leq ct^2 (1+\lambda) \parallel u \parallel_{L^\infty(K)},$$

hence

$$| I_5 - \frac{1}{h_+} \frac{p_2'(x_0)}{2\sqrt{p_2(x_0)}} \int_0^{\varrho_2(h_+)} u'(x_0 + 0) t dt | \leq$$

$$\leq c(1+\lambda)\bar{h}^2 \parallel u \parallel_{L^\infty(K)},$$

and analogously

$$| \hat{I}_5 - \frac{1}{h_-} \frac{p_1'(x_0)}{2\sqrt{p_1(x_0)}} \int_0^{\varrho_1(h_-)} u'(x_0 - 0) t dt | \leq$$

$$\leq c(1+\lambda)\bar{h}^2 \parallel u \parallel_{L^\infty(K)} .$$

We have

$$h_+ = \int_0^{\varrho_2(h_+)} \frac{1}{\sqrt{p_2}} = \frac{\varrho_2(h_+)}{\sqrt{p_2(x_0)}} + \underline{\underline{O}}(\bar{h}^2),$$

$$\frac{1}{2}[\varrho_2(h_+)]^2 = \frac{h_+^2}{2} p_2(x_0) + \underline{\underline{O}}(\bar{h}^3),$$

consequently

$$| I_5 - h_+ \frac{p_2'(x_0)}{4\sqrt{p_2(x_0)}} p_2(x_0) u'(x_0 + 0) | \leq c(1+\lambda)\bar{h}^2 \parallel u \parallel_{L^\infty(K)}$$

and analogously

$$| \hat{I}_5 - h_- \frac{p_1'(x_0)}{4\sqrt{p_1(x_0)}} p_1(x_0) u'(x_0 - 0) | \leq c(1+\Lambda)\bar{h}^2 \parallel u \parallel_{L^\infty(K)}$$

so

$$| I_5 + \hat{I}_5 - p_2(x_0) u'(x_0 + 0) \left[h_+ \frac{p_2'(x_0)}{4\sqrt{p_2(x_0)}} + h_- \frac{p_1'(x_0)}{2\sqrt{p_1(x_0)}} \right] | \leq$$

$$\leq c(1+\lambda)\bar{h}^2 \parallel u \parallel_{L^\infty(K)}$$

holds if (17) is true. Hence we have to choose the pairs h_+, h_- such hat that (21) holds and this is ensured whenever $p_2'(x_0) p_1'(x_0) < 0$. The above estimates prove Theorem 3.

2. Inhomogeneous Sturm-Liouville difference equation

$$-\mathbf{u}'' + \mathbf{q}\mathbf{u} = \mathbf{f}$$

with boundary conditions.

2.1 Difference approximation for equidistant nodes. Consider the following boundary value problem

$$-u_n'' + qu_n = \lambda_n u_n \qquad 0 < x < 1$$

$$u_n(0) = u_n(1) = 0 \tag{22}$$

with $q \in C[0,1]$. It is known that in this case the eigenfunctions $(u_n)_{n=1}^{\infty}$ form complete othonormal system in $L^2(0,1)$, the eigenvalues λ_n are real and tend to $+\infty$. If we add a positive constant to q we can ensure that

$$\lambda_n \geq 1;$$

we shall suppose it. Consider further the equation

$$-u'' + qu = f$$

$$u(0) = u(1) = 0 \tag{23}$$

and its difference analogon

$$-y_{\bar{x}x} + dy = f, x \in \omega_h := \{x_i = ih, 0 < i < N\} \tag{24}$$

$$y_0 = y_N, \quad h = \frac{1}{N}$$

where $y_{\bar{x}x}$ and d are defined as in Theorem 1. We shall prove

Theorem 4

Let $f \in C^2[0,1], -f'' + qf \in C^2[0,1], 0 = f(0) = f(1) = (-f'' + qf)(0) = (-f'' + qf)(1)$. Define the operators $Lu : -u'' + qu - f, \Lambda u := -u_{xx} + du - f$. Now if u, u' are absolutely continuous and $u(0) = u(1) = 0$, then

$$Lu = 0 \Rightarrow max_{x \in \omega_h} \mid \Lambda u(x) \mid = \underline{\underline{O}}(h).$$

If, in additon, $q \in Lip1$ then

$$Lu = 0 \Rightarrow max_{x \in \omega_h} \mid \Lambda u(x) \mid = \underline{\underline{O}}(h^2).$$

<u>Proof</u>

Condsider the expansion of $f(x)$:

$$f = \sum_{n=1}^{\infty} c_n u_n.$$

Here

$$c_n = \langle f, u_n \rangle = \langle f, \frac{-u_n'' + qu_n}{\lambda_n} \rangle = \langle -f'' + qf, \frac{u_n}{\lambda_n} \rangle.$$

It is known from the spectral theory of the Schrödinger operators [16] that the expansion of $-f'' + qf$ is uniformly convergent on [0,1], hence

$$\sum_{n=1}^{\infty} \lambda_n \mid c_n \mid < \infty \mid . \tag{25}$$

Define the function

$$u := \sum_{n=1}^{\infty} \frac{c_n}{\lambda_n} u_n.$$

By (25) this sum converges uniformly (cf.(4)), hence $u(0) = u(1) = 0$. The series

$$\sum_{n=1}^{\infty} \frac{c_n}{\lambda_n} u_n''$$

converges also uniformly and twofold derivation gives that

$$u'' = \sum_{n=1}^{\infty} \frac{c_n}{\lambda_n} u_n''$$

and hence u is the /unique/ solution of (23). Now

$$\Lambda u(x) = -u_{\bar{x}x} + du - f = -\sum_{n=1}^{\infty} \frac{c_n}{\lambda_n} (u_n)_{\bar{x}x} +$$

$$+\sum_{n=1}^{\infty} \frac{c_n}{\lambda_n} u_n(x) d(x) - \sum_{n=1}^{\infty} c_n u_n(x) =$$

$$= \sum_{n=1}^{\infty} \frac{c_n}{\lambda_n} \left[-(u_n)_{\bar{x}x} + d(x) u_n(x) - \lambda_n u_n(x) \right].$$

We apply here the estimates given in proving Theorem 1 to obtain

$$\max_{x \in \omega_h} | \Lambda u(x) | \leq c \sum_{n=1}^{\infty} \frac{c_n}{\lambda_n} (\lambda_n^2 h^2 + \sqrt{\lambda_n} h) \leq ch.$$

If $q \in Lip1$ then

$$\max_{x \in \omega_h} | \Lambda u(x) | \leq c \sum_{n=1}^{\infty} \frac{c_n}{\lambda_n} \lambda_n^2 h^2 \leq ch^2.$$

Theorem 4 is proved.

<u>**Remark**</u>

We can also formulate the analogon of Theorem 4 for non-equidistant point system using the estimates of Theorem 2.

2.2 Uniform convergence of the solution of the difference equation (24) to the solution of (23).

In this point we consider the solution $y = y_h$ of (24). Define

$$y_x := \frac{y(x+h) - y(x)}{h}.$$

We need the following results, representing the difference analogon of some inclusion theorems of S.L.Soboleff /[8],p. 118, 120./

Lemma 3

If a function y is given on the set

$$\bar{\omega}_h := \{x_i = ih : 0 \le i \le N\}$$

and $y(0) = y(1) = 0$ then

$$\| y \|_c \le \frac{1}{2} \| y_x] |$$

where

$$\| y \|_c := \max_{x \in \bar{\omega}_h} | y(x) |,$$

$$\| y_x] | := (y_x, y_x]^{\frac{1}{2}} := \left(\sum_{i=1}^{N} h y_x^2(x_i) \right)^{\frac{1}{2}}.$$

Lemma 4

If the assumptions of Lemma 3 hold, then

$$\frac{h^2}{4} \| y_x] |^2 \le \| y \|^2 \le \frac{1}{8} \| y_x] |^2$$

where

$$\| y \| := \left(\sum_{i=1}^{N-1} y_i^2 h \right)^{1/2}, y_i := y_x(x_i).$$

Now multiply the equality in (24) by $y_i h$ and sum up over ω_h, then we get

$$\sum_{i=1}^{N-1} (y_{xx})_i y_i h - \sum_{i=1}^{N-1} d_i y_i^2 h + \sum_{i=1}^{N-1} f_i y_i h = 0,$$

or, writing scalar products

$$(y_{xx}, y) - (d, y^2) + (f, y) = 0.$$

Using the difference analogon of the first Green formula /[8], p.110/ we get

$$-(y_x, y_x] - (d, y^2) + (f, y) = 0$$

i.e.

$$\| y_x] |^2 + (d, y^2) = (f, y).$$

Suppose that

$$d_i \geq 0, \quad i = 1, \dots, N-1.$$

Then by the Cauchy-Schwartz inequality we obtain

$$\| y_x] |^2 \leq \| f \| \| y \| .$$

By Lemmas 3. and 4. we get

$$4\sqrt{2} \| y \| \| y \|_c \leq \| y_x] |^2$$

and hence the a priori estimate

$$\| y \|_c \leq \frac{1}{4\sqrt{2}} \| f \| \tag{26}$$

holds. Now take the solution u of (23) and consider the difference

$$z := u - y.$$

It obviously satisfies

$$z_{\bar{x}x} - dz = u_{\bar{x}x} - du - y_{\bar{x}x} + dy = u_{\bar{x}x} - du + f = -\Lambda u$$

by the terminology of Theorem 4. The estimate (26) and Theorem 4 give that

$$\| z \|_c \leq \frac{1}{4\sqrt{2}} \| \Lambda u \| = \underline{\underline{O}}(h).$$

So the following statement is proved.

Theorem 5

Suppose the conditions given in Theorem 4 further let $q(x) \geq 0, x \in [0, 1]$. Take the solution u of (23) and $y = y_h$ of (24). Then

$$\| u - y_h \|_c = \underline{\underline{O}}(h).$$

Remark

By Lemma 4 the operator

$$Ay := -y_{\bar{x}x} + dy \qquad 0 \leq d \leq c$$

is positive definite, namely

$$(Ay, y) \geq (y_{\bar{x}}, y_{\bar{x}}] = \| y_{\bar{x}}] |^2 \geq 8 \| y \|^2 .$$

Further A is also selfadjoint by the second Green formula /[8], p. 110/.

<u>Remark</u> On the application of the Taylor formula.

When approximating differential operators by difference operators, it is useful to use Taylor formulas /[8]/. This method requires that the functions from the domain of the operators be very smooth. E.g. for the differential operator $Lu = u''$ it is known that

$$u'' - u_{\bar{x}x} = \underline{\underline{O}}(h^2) \tag{27}$$

whenever $u \in C^4[x - h_0, x + h_0]$ and x and h_0 are fixed ([8], p. 75). By a modified use of the Taylor formula we can prove the following proposition:

Let

$$u \in C^3[x - h_0, x + h_0], \quad u^{(3)} \in Lip\alpha \text{ on } [x - h_0, x + h_0].$$

Then

$$u'' - u_{\bar{x}x} = \underline{\underline{O}}(h^{1+\alpha}).$$

If $u^{(3)} \in Lip1$, then we get (27).

Observe that Theorem 1 gives (27) under more weaker smoothness condition. To prove the proposition, write the Taylor formulas consisting of four members

$$u(x + h) = u(x) + u'(x)h + u''(x)\frac{h^2}{2} + u^{(3)}(\xi_+)\frac{h^3}{6},$$

$$u(x - h) = u(x) - u'(x)h + u''(x)\frac{h^2}{2} - u^{(3)}(\xi_-)\frac{h^3}{6}$$

with some

$$\xi_+ \in (x, x+h), \quad \xi_- \in (x-h, x).$$

This gives

$$u_{xx} - u''(x) = h\frac{u^{(3)}(\xi_+) - u^{(3)}(\xi_-)}{6}$$

which proves at once the proposition.

We can analogously investigate the case of non-equidistant nodes where $u_{x\hat{x}}$ is defined in (14). We can assert:

Let

$$u \in C^2 \text{ and } u'' \in Lip\alpha \text{ for some } 0 < \alpha \leq 1.$$

Then

$$u'' - u_{x\hat{x}} = \underline{\underline{O}}(\bar{h}^\alpha).$$

Indeed, by the three member Taylor formulas

$$u(x+h_+) = u(x) + h_+ u'(x) + \frac{h_+^2}{2} u''(\xi_+), \quad \xi_+ \in (x, x+h_+),$$

$$u(x-h_-) = u(x) - h_- u'(x) + \frac{h_-^2}{2} u''(\xi_-), \quad \xi_- \in (x-h_-, x)$$

we get

$$u_{x\hat{x}} = \frac{2}{h_+ + h_-}\left[\frac{u(x+h_+) - u(x)}{h_+} + \frac{u(x-h_-) - u(x)}{h_-}\right] =$$

$$= \frac{h_+ u''(\xi_+) + h_- u''(\xi_-)}{h_+ + h_-} = u''(\xi), \quad \xi \in (\xi_-, \xi_+)$$

hence

$$u_{x\hat{x}} - u''(x) = u''(\xi) - u''(x) = \underline{\underline{O}}(\bar{h}^\alpha)$$

as we asserted.

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