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ON DOMAINS OF ATTRACTION OF EXTREME VALUE DISTRIBUTIONS VIA GENERALIZED CONCAVITY - CONVEXITY

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1. Introduction

Consider a sequence of independent, identically distributed random variables $X_1, X_2, X_3 \ldots$ with common distribution function (d.f) $F(x)$. For $n \geq 1$, let

$$Z_n = max(X_1, X_2, \ldots, X_n).$$

A d.f. F belongs to the domain of attraction of a nondegenerate d.f. H, if there exist sequences of constants $\{a_n, b_n\}_{n=1}^{\infty}$ with $b_n > 0$, such that

(1.1)
$$\lim_{n \to \infty} P(Z_n < a_n + b_n x) = H(x)$$

holds at all continuity points of H. In the sequel the relation (1.1) will be denoted by $F \in D(H)$. We eploy the notation
$H_{1,\gamma}(x) = exp(-x^{-\gamma}), x > 0, H_{2,\gamma}(x) = exp(-(-x)^{\gamma}), x < 0$ and $H_{3,0}(x) = exp(-e^{-x}), x \in R$, where γ is a positive parameter.
A d.f. F can belong only to the domain of attraction of one of the three types $H_{1,\gamma}, H_{2,\gamma}$ and $H_{3,0}$ (see e.g. [1]). Gnedenko gave characterizations of domains of attraction of the three types in [2]. De Haan presented a unified approach involving only the d.f. F itself (see [3]).

The aim of the present paper is to characterize the domains of attraction of extreme value distributions in terms of the concavity-convexity index of the tail of an appropriate d.f. G, or equivalently: in terms of ultimately concavity (convexity) properties of the function $[1 - G(x)]^q (q \in R, q \neq 0)$.

In section 2 we give the concepts of r-concave (r-convex), ultimately r-concave (ultimately r-convex) functions and we investigate the main properties of these functions (from our point of wiew). Moreover, we define concavity, convexity and concavity-convexity indexes of a function g, denoted by $cv(g), cx(g)$ and $c(g)$, respectively. In the case of twice differentiable g, we can give an alternative definition of $c(g)$.

This last result leads us (in Section 3) to the above-mentioned characterizations being supported by the paper [3].

2. Generalized concave and convex functions

The notion of convexity and concavity undoubtedly plays a dominant role in great many branches of mathematics. There exist some different generalizations of these concepts. The definition of r-concave (r-convex) functions is due to Martos (see [4]) and generalizes the usual concavity (convexity) by letting the weighted artithmetic mean of funcion values be replaced by a more general mean.

Let u,v be positive real numbers, $\lambda \in (0,1)$ and let $r \neq 0$ be a real number. The λ-weighted r-meand of u,v is defined by

(2.1)
$$M_\lambda{}^r(u,v) = [\lambda u^r + (1-\lambda)v^r]^{1/r}.$$

Applying continuity argumentation, one can easily see that

$$M_\lambda^0 \ (u,v) = u^\lambda v^{1-\lambda},$$
$$M_\lambda^{+\infty} (u,v) = max(u,v),$$
$$M_\lambda^{-\infty} (u,v) = min(u,v).$$

A positive real function g(x) defined on some interval (a,b) is said to be an r-concave function on (a,b) , if for any pair $x_1, x_2 \in (a,b)$ and $\lambda \in (0,1)$ we have the following inequality

(2.2)
$$g(\lambda x_1 + (1-\lambda)x_2) \geq M_\lambda^r(g(x_1), g(x_2)),$$

where $-\infty \leq a < b \leq +\infty$.

Note that r-convex functions are defined in a similar way, with the opposite sign inequality.

A simple characterization of r-concave (r-convex) functions can be obtained by the familiar notion of ordinary concavity and convexity:

a.) *$g(x)$ is r-concave (r-convex) if and only if*

(i) *$g^r(x)$ is concave (convex) when $r > 0$;*

(ii) *$g^r(x)$ is convex (concave) when $r < 0$;*

(iii) *$\log g(x)$ is concave (convex) when $r = 0$.*

b.) *If $g(x)$ is r-concave (r-convex), then it is also q-concave (q-convex) for every $q < r (q > r)$.*

In case of differentiable functions alternative definitions of r-concave and r-convex functions can be given. Let g(x) be a twice differentiable positive function on an open interval (a,b). Denote by $g'(x), g''(x)$ the first and second derivatives of g(x), respectively. Then g(x) is r-concave on (a,b) if and only if the inequality

(2.3)
$$(r-1)[g'(x)]^2 + g(x)g''(x) \leq 0$$

holds for every $x \in (a,b)$.

We note that inequality (2.3) is reversed when r-convex functions are characterized.

Assume that g(x) is a real, non-negative function on some interval (a,b), where $-\infty \leq a < b \leq +\infty$, and let

$$\omega(g) = sup\{x : g(x) > 0\}.$$

g(x) is said to be an ultimately r-concave (ultimately r-convex) function if for the given $r \in R$ there exists an x_r such that $x_r < \omega(g)$ and $g(x)$ is r-concave (r-convex) on the interval $(x_r, \omega(g))$. Let us define the concavity and convexity indexes of $g(x)$ by
(2.4)

$$cv(g) = \sup\ \{r \in R : g(x) \text{ is ultimately r} - \text{concave}\}$$

and
(2.5)

$$cx(g) = \inf\ \{r \in R : g(x) \text{ is ultimately r} - \text{convex}\},$$

respectively.
These terms are well-defined because of the part b.) of the above-mentioned characterization.
If for a given $g(x)$ we have the equality $cv(g) = cx(g)$ then let the concavity-convexity index (or briefly : cc-index) of g be $c(g) = cv(g) = cx(g)$. The following result is fundamental from our point of view.

Proposition. *Assume that for a given positive function g(x) there exists an* $\underline{x} < \omega(g)$ *such that g is twice differentiable and* $g'(x) \neq 0$ *on the interval* $(\underline{x}, \omega(g))$. *Then* $cv(g) = cx(g) = q$ *if and only if the following relation holds:*
(2.6)

$$\lim_{x \to \omega} \left[\frac{g(x)}{g'(x)}\right]' = q \cdot \sqcap$$

Proof. Assume that $c(g) = q$. This means that for every $\epsilon > 0$ $g(x)$ is ultimately $(q-\epsilon)$-concave and is ultimately $(q+\epsilon)$-convex. It follows from the inequality (2.3) that we have

(2.7)
$$q - \epsilon \leq 1 - \frac{g(x)g''(x)}{[g(x)]^2} \leq q + \epsilon$$

on some interval $(x(\epsilon), \omega(g))$, whence we get the relation (2.6). The converse is obvious. Thus the theorem is proved. ☐

3. Domains of attraction results

Let F(x) be a d.f.. Assume that there exits an $x_F < \omega = \omega(1 - F)$ such that

(3.1)
$$\int_{x_F}^{\omega} [1 - F(t)]dt < +\infty.$$

In this case let us define the $d.f.$'s $F_1(x)$ and $F_2(x)$ by

(3.2)
$$F_1(x) = max\{0, \int_{x}^{\omega} [1 - F(t)]dt\}$$

(3.3)
$$F_2(x) = max\{0, \int_{x}^{\omega} [1 - F_1(t)]dt\}.$$

We note that (3.1) holds trivially when $\omega < +\infty$. If the integral in (3.1) is not bounded above we can define a new $d.f. F^*(x)$ by

(3.4)
$$F^*(x) = 1 - \frac{1 - F(x)}{x^3}, x \geq 1.$$

De Haan proved in [3] that (3.1) is valid with $F^*(x)$ instead of $F(x)$. So in the case $\omega = +\infty$ let F_1^* and F_2^* be defined by (3.2) and (3.3), respectively, with F^* instead of F.

Now we are able to draw up our main results.

Theorem.

a.) $F \in D(H_{1,\gamma})$ *if and only* $\omega = \infty$ *and there exists an* r_1 *such that* $-1 < r_1 < 0$ *and the function* $[1 - F_2^*(x)]^r$ *is ultimately concave for* $r_1 < r < 0$ *and ultimately convex for* $r < r_1$ *or* $r > 0$. *Moreover,* $\gamma = -1/r_1 - 1$.

b.) $F \in D(H_{2,\gamma})$ *if and only if* $\omega < \infty$ *and there exists an* r_0 *such that* $0 < r_0 < 1/2$ *and the function* $[1 - F_2(x)]^r$ *is ultimately concave for* $0 < r < r_0$ *and is ultimately convex for* $r < 0$ or $r > r_0$.
Moreover, $\gamma = 1/_{r_0} - 2$.

c.) $F \in D(H_{3,0})$ *if and only if the function* $[1 - F_2(x)]^r$ *is ultimately convex for every* $r \in R, r \neq 0$.

Proof. De Haan proved that $F \in D(H_{l,\gamma})$ *iff* $\omega = \infty$ and

$$\lim_{x \to \infty} \frac{[1 - F^*(x)][1 - F_2^*(x)]}{[1 - F_1^*(x)]^2} = d \qquad \text{with } 1 < d < 2$$

$$\text{and} \quad \gamma = \frac{2 - d}{d - 1} \quad \text{(see Theorem 12 in [3]).}$$

It is easy to see by our Proposition that this is equivalent to $c(1 - F_2^*) = 1 - d, i.e., 1 - F_2^*(x)$ is ultimately r-concave for $r < r_1 = 1 - d$ and ultimately s-convex for $s > r_1$. The proof is complete if we apply the caracterizations by ordinary concavity (convexity).

b.) In the above-mentioned paper of de Haan [3] the Theorem 11 asserts that $F \in D(H_{2,\gamma})$ $iff \omega < \infty$ and

(3.5)

$$\lim_{x \uparrow \omega} \frac{[1 - F(x)][1 - F_2(x)]}{[1 - F_1(x)]^2} = d \text{ with } 1/2 < d < 1$$

and

$$\gamma = \frac{2d-1}{1-d},$$

and by the proposition, (3.5) is rue *iff* $c(1-F_2) = 1-d, 0 < 1-d < 1/2$, *i.e.*, $1-F_2(x)$ is ultimately r-concave for $r < r_0 = 1-d$ and ultimately s-convex for $s > 0$. One may now finish the proof as in part a.).

c.) In [3] it was proved also that $F \in D(H_{3,0})$ iff (3.5) holds with d=1 (see Theorem 10 in [3]). This is equivalent to $c(1-F_2) = 0$, *i.e.*, $1-F_2(x)$ is ultimately r-concave for every $r < 0$ and ultimately s-convex for $s > 0$. Thus our theorem is proved.

References

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