Annales Univ. Sci. Budapest., Sect. Comp. 10 (1989) 35-44

A COMPUTER-ASSISTED NUMBER THEORETICAL CONSTRUCTION OF (3,K)-RAMSEY GRAPHS

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Abstract : The concept of maximal triangle-free circular graphs is introduced and their relation to sum-free bases for finite intervals of integers is studied, leading to the construction of an infinite sequence of (3, k)-Ramsey graphs if $k \geq 61$.

1. Introduction

Throughout, graphs are undirected, with no loops or multiple edges. The number of vertices will be denoted by n, the degree of a vertex v by $d(v)$, the distance between vertices u, v by $d(u, v)$. A set of vertices is independent (or stable) if the vertices are pairwise non-adjacent. The maximum size of such a set in a graph G is called the independence (or stability) number and will be denoted by $\alpha\ (G)$. For a vertex v, $N(v)$ will denote the set of its neighbours. graph is triangle-free if it does not contain three mutually adjacent vertices.

We shall use the symbol $gcd(a_1, a_2, \ldots)$ for the greatest common divisor of the integers $a_1, a_2, \ldots$. $[n]$ will denote the integer part of n.

The Ramsey number $R(3, k)$ is the smallest integer n such that any graph with n vertices either contains a triangle K_3 or an independent set of size k. The asymptotic bounds

$$\frac{ck^2}{log^2 k} < R(3,k) < \frac{ck^2}{log k}$$

are known since many years, the lower bound is due to Erdős [1] and the upper bound to Ajtai, Komlós and Szemerédi [2] with c=100. This constant was improved to 2.4 by Griggs [3, 4]. A graph is called a ***(3, k)-Ramsey graph*** if it contains neither a triangle, nor a $k-$ *element* independent set. Constructing such graphs can lead to improve lower bounds for $R(3,k)$.

2. Circular and maximal triangle-free circular graphs

Let $1 \leq j_1 < j_2 \ldots < j_k < [\frac{n}{2}]$ be integers. We define $G = G(n; j_1, j_2, \ldots, j_k)$ as a simple undirected graph with n vertices where two vertices are adjacent if and only if the difference of their indices equals one of $j_1, j_2, \ldots, j_k$ (modulo n). Such graphs are called circular. For example, $G(5;1)$ is a circuit of length 5.

A maximal triangle-free circular graph (**MTC**) is a triangle-free circular graph G with $G+e$ containing triangles for any edge e not belonging to G.

An infinite sequence $G_3, G_4, \ldots$ of **MTC**'s can be obtained if G_3 is a circuit of length 5 and G_{r+1} is obtained from G_r by associating a new vertex x' with each vertex x of G_r and joining it to all neighbours of x in G_r; moreover, by adding a new vertex y and joining it to the new vertices x', see [5].

One can easily prove the following statements:

(1). Circular graphs are regular.

(2). If $G(n; j_1, j_2, \ldots, j_k)$ with $k \geq 2$ is a triangle-free circular graph then its girth is 4.

(3). Let $k = gcd(n,j)$ and $h = \frac{n}{k}$ for a circular graph $G(n,j)$. Then the graph consists of k circuits of length h [7].

(4). If $gcd(n,j_1,j_2,\dots,j_k) = 1$ for a circular graph $G(n;j_1,j_2,\dots,j_k)$ then the graph is connected [7].

(5). A triangle-free graph G is maximal with respect to this property if and only if the distance of any two vertices of G is at most 2.

(6). If G is a maximal triangle-free graph and v is a vertex of G with maximum degree Δ then $N(v)$ is a maximal independent set of G.

(7). If G is a **MTC** then $N(v)$ is a maximal independent set of G for every vertex v of G.

(8). A circular graph $G(n;j_1,j_2,\dots,j_k)$ is triangle-free if and only if

(i). $j_{i_1} + j_{i_2} + j_{i_3} \neq n$ for any three, not necessarily distinct integers $j_{i_1}, j_{i_2}, j_{i_3} \in \{j_1,j_2,\dots,j_k\}$;

(ii). $j_{i_1} + j_{i_2} \neq j_{i_3}$ for any three integers $j_{i_1}, j_{i_2}, j_{i_3} \in \{j_1,j_2,\dots,j_k\}$ (where j_{i_1} and j_{i_2} can be equal).

(The necessity of these conditions is obvious. For the sufficiency suppose that the graph has a triangle T. Without loss of generality we may suppose that the vertices of T contain the n^{th} vertex. Let the subscript of the other two vertices be h and k with $h < k$. If $k \leq [\frac{n}{2}]$ then the choice $h = j_{i_1}$ and $k = j_{i_2}$ leads to a contradiction with (ii). Otherwise put $h = j_{i_1}$ and $k - h = j_{i_2}$ for a contradiction with (i).)

(9). Let $G(n;j_1,j_2,\dots,j_k)$ be a triangle-free circular graph, let $S = \{j_1,j_2,\dots,j_k\}$. Suppose that S is a base for the interval $[1,[\frac{n}{2}]]$ and that for any $j \in S$ even, there are some $j_{i_1}, j_{i_2} \in S$ so that $j_{i_1} + j_{i_2} = \frac{j}{2}$ or $|j_{i_1} - j_{i_2}| = \frac{j}{2}$. Then the graph is **MTC**.

(The statement easily follows from statement (8). The conditions are also necessary. For example, $G(10;2,3)$ and $G(10;1,4)$ are **MTC** while $G(10;2,5)$ is not.)

We associate a $(2k+1)\times 2k$ matrix M_v to each vertex v of a circular graph $G(n;j_1,j_2,\ldots,j_k)$ as follows. The first row $A_v=(a^v_{1,1},a^v_{1,2},\ldots,a^v_{1,2k})$ consists of the elements $a^v_{1,i}\in N(v)$ in circular order (starting with $v+j_1$, ending with $v+(n-j_1)$ and increasing mod n). The elements of $N(a^v_{1,i})$ in the same order are arranged as column vectors $b_1,b_2,\ldots,b_{2k}$ for every $i=1,2,\ldots,2k$. These vectors form a $2k\times 2k$ block B_v which is placed under A_v.

For example, if the vertices of the **MTC** graph $G(65;7,10,11,12,13,15,16)$ are numbered from 0 to 64 then M_0 is shown on the next page.

The following statements are again straightforward:

(10). For any vertex v of a circular graph $G(n;j_1,j_2,\ldots,j_k)$, the matrix B_v has the following properties:

(i). B_v is symmetric;

(ii). $b_{i,i}=2a^v_{1,i}\pmod n-v$;

(iii). $b_{1,2k}=b_{2,2k-1}=\ldots=b_{2k,1}=v$;

(iv). $b_{i,j}+b_{2k-j+1,2k-i+1}=2v\pmod n$ for every i, j =1, 2, ... , 2k.

(11). For the first vertex of $G(n;j_1,j_2,\ldots,j_k)$, i.e. for that with label 0,

(i). $A_0=(j_1,j_2,\ldots,j_k,n-j_k,\ldots,n-j_1)$;

(ii). $b_i^T=A_{j_i}$ and $b_{k+i}^T=A_{n-j_{k+1-i}}$ for every i =1, 2, ... , k;

(12). Let the vertices of the circular graph $G(n;j_1,j_2,\ldots,j_k)$ be labeled from 0 to n-1. If there are subsets $S_1\subseteq N(0)$ and

$S_2 \subseteq N(k)$ where $k \in N(0)$ so that $S = \{N(0) - S_1\} \cup S_2$ is also an independent set then $|S_2| \leq |S_1|$, i.e. $|S| \leq |N(0)|$.

MAT M_0

MAT A_0 07...10...11...12...13...15...16...49...50...52...53...54...55...58

MAT B_0 14...17...18...19...20...22...23...56...57...59...60...61...62...00

17...20...21...22...23...25...26...59...60...62...63...64...00...03

18...21...22...23...24...26...27...60...61...63...64...00...01...04

19...22...23...24...25...27...28...61...62...64...00...01...02...05

20...23...24...25...26...28...29...62...63...00...01...02...03...06

22...25...26...27...28...30...31...64...00...02...03...04...05...08

23...26...27...28...29...31...32...00...01...03...04...05...06...09

56...59...60...61...62...64...00...33...34...36...37...38...39...42

57...60...61...62...63...00...01...34...35...37...38...39...40...43

59...62...63...64...00...02...03...36...37...39...40...41...42...45

60...63...64...00...01...03...04...37...38...40...41...42...43...46

61...64...00...01...02...04...05...38...39...41...42...43...44...47

62...00...01...02...03...05...06...39...40...42...43...44...45...48

00...03...04...05...06...08...09...42...43...45...46...47...48...51

3. Sum-free bases and MTC graphs

Let S be a set of integers. $S+S$ and $S-S$ denote the sets of integers arising as sums, or differences, respectively, of two elements of S. S is called sum-free if $S \cap (S+S) = S \cap (S-S) = \phi$.

A subset S of $\{1, 2, \ldots, n\}$ is called a base if $S \cup (S+S) \cup (S-$

$S) = \{1, 2, \ldots, n\}$. The minimum cardinality c_n of such a base was proved to be $c_n \le \sqrt{3.6}\sqrt{n}$, see [8]; this was later improved to be $c_n \le \sqrt{3.5}\sqrt{n} + o(\sqrt{n})$, see [9].

The constructions in [8] and [9] are not suitable for our purposes since these bases are not sum-free. In this section we construct a sum-free base (with a somewhat larger cardinality).

Theorem 1: Let $t \ge 4$ and $n = 10t^2 + 19t + 3$. The subset $S \subseteq [1, 2, \ldots, n]$ of integers be the union of the following seven arithmetic progressions:

I. $\alpha_1 = 2t + 1$, $\alpha_2 = 3t + 2$, ... , $\alpha_{3t} = 3t^2 + 4t$; here the difference of the consecutive terms is $d_1 = t + 1$.

II. $\beta_1 = 3t^2 + 4t + 2$, $\beta_2 = 3t^2 + 4t + 3$, ... , $\beta_{t-2} = 3t^2 + 5t - 1$; $d_2 = 1$.

III. $\gamma_1 = 3t^2 + 5t + 1$, $\gamma_2 = 3t^2 + 5t + 2$, ... , $\gamma_t = 3t^2 + 6t$; $d_3 = 1$.

IV. $\delta_1 = 3t^2 + 6t + 2$.

V. $\epsilon_1 = 6t^2 + 12t + 3$, $\epsilon_2 = 6t^2 + 13t + 3$, ... , $\epsilon_{t+2} = 7t^2 + 13t + 3$; $d_5 = t$.

VI. $\eta_1 = 6t^2 + 13t + 4$, $\eta_2 = 6t^2 + 14t + 5$, ... , $\eta_{t-4} = 7t^2 + 9t - 1$; $d_6 = t + 1$.

VII. $\lambda_1 = 7t^2 + 11t + 1$.

Then

(1). S has cardinality $7t - 2$;

(2). $S \cup (S + S) \cup (S - S)$ covers every integer between 1 and n except $7t^2 + 10t$ and $7t^2 + 12t + 2$;

(3). S is sum-free;

(4). $a+b+c \neq 2n$ for any $a, b, c \in S$.

The first statement is obvious. (2) can be verified by the following sequence of statements:

1. 1 and 2 can be covered easily.

2. $\{\gamma_i - \beta_j\} \supseteq [3, t+1]$.

3. $\{\alpha_i\} \cup \{\gamma_i - \alpha_j\} \cup \{\delta_1 - \alpha_i\} \supseteq [t+2, 3t^2+4t+1]$.

4. $\{\beta_i\} \supseteq [3t^2+4t+2, 3t^2+5t-1]$.

5. $\alpha_1 + \alpha_{3t-1} = 3t^2+5t$.

6. $\{\gamma_i\} \supseteq [3t^2+5t+1, 3t^2+6t]$.

7. $\alpha_1 + \alpha_{3t} = 3t^2+6t+1$.

8. $\delta_1 = 3t^2+6t+2$.

9. $\{\epsilon_i - \gamma_j\} \supseteq [3t^2+6t+3, 4t^2+8t+2]$.

10. $\{\epsilon_i\} \cup \{\epsilon_i - \alpha_j\} \cup \{\eta_i\} \cup \{\alpha_i + \epsilon_j\} \supseteq [4t^2+8t+3, 7t^2+10t+3]$, except $7t^2+10t$.

11. $\{\alpha_i + \epsilon_j\} \supseteq [7t^2+10t+4, 7t^2+11t]$.

12. $\lambda_1 = 7t^2+11t+1$.

13. $\epsilon_{t+2} - \alpha_1 = 7t^2+11t+2$.

14. $\{\epsilon_i\} \cup \{\alpha_i + \epsilon_j\} \supseteq [7t^2+11t+3, 9t^2+18t+3]$, except $7t^2+12t+2$.

15. $\{\gamma_i + \epsilon_j\} \supseteq [9t^2+18t+3, 10t^2+19t+3]$.

(In order to see item no. 10, observe that if $t=4$, $\{\epsilon_i\} \cup \{\epsilon_i - \alpha_j\} \supseteq [4t^2+8t+3, 7t^2+10t+3]$ except $7t^2+10t$ and that in case of $t \geq 5$ we also need $\{\alpha_i + \epsilon_j\}$, i, j =1, 2, ..., t-3, for the numbers of form $[7t^2 + (10 - i)t + 3 + j]$, i =1, 2, ... , t-4; j =1, 2, ... , (t-4)+1-i, and then only $7t^2 + 10t$ and the η_i's are left. The other items are obvious.)

The proofs of (3) and (4) require a large number of technical steps. The interested reader is referred to [12]; details in English are available from the author.

Using the above sets $S = \{j_1, j_2, \ldots, j_k\}$ we obtain an infinite sequence of **MTC** graphs $G(n; j_1, j_2, \ldots, j_k)$, i.e. an infinite sequence of $(3, k)$ – ***Ramsey*** graphs.

4. Computational results and some remarks

Using some ideas of Balas and Chang [10] we developed a **FORTRAN program** for finding a maximum independent set in a circular graph, and determined the first eight **MTC** graphs of the above sequence (for t =4, 5, 6, 7, 8, 9, 10, and 11). The main parameters of these graphs are as follows:

t	n	Δ	α
4	478	52	60
5	696	66	82
6	954	80	102
7	1252	94	160
8	1590	108	187
9	1968	122	255
10	2386	136	239
11	2844	150	300

The explicit lists of S and a maximum independent set for each graph can be found in [12] and are available from the author.

Some open problems arise from these considerations. For example, we conjecture that $\frac{\alpha}{n}$ is increasing if t is odd and decreasing if t is even for the above sequence of **MTC** graphs.

Acknowledgement : My sincere thanks are due to Prof. L. Lovász and A. Recski for numerous valuable advices. Also all cordial helps from M. Janos, Susan, Tomi, Laci, Imre, and Gy. Elekes are unforgettable.

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