

# AN ITERATIVE METHOD FOR NONSELFADJOINT ELLIPTIC PROBLEMS ON REGIONS PARTITIONED INTO SUBSTRUCTURES

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**Abstract.** A new preconditioned conjugate gradient (PCG)-based domain decomposition method is given for the solution of linear equations arising in the finite element method applied to a nonselfadjoint elliptic problem. The domain under consideration is broken into subdomains and the preconditioner is defined such that it requires only the solution of matrix problems on the subdomains. Analytic estimates are given which under appropriate hypotheses guarantee the geometric convergence of the preconditioned iterative procedure. The rate of convergence is independent of the number of unknowns.

## 1. Introduction

In the paper we describe a PCG-based domain decomposition method for the solution of linear equations which arises from the discretization of second-order nonselfadjoint uniformly elliptic boundary value problem in a bounded region  $\Omega$  via finite element method. For the sake of exposition we will assume that  $\Omega \subset R^2$  and it is partitioned into two subregions  $\Omega_1$  and  $\Omega_2$ .

Domain decomposition methods for elliptic problems are described in many papers see [1-2], [5], [7-8] and the references given there in. Our method is based on technique explained in [1], [5] and [7]. Here we give an extension of method of decomposition for selfadjoint elliptic problems proposed in [1] for the nonselfadjoint elliptic problems.

The nonselfadjoint case is studied for example in [5] using the minimum residual method which is less effective than the method proposed here.

## 2. Differential and Discrete Problems

We consider as a model problem the weak form of the following Dirichlet problem for the second order elliptic equation.

For  $f \in L^2(\Omega)$  find a function  $u \in H_0^1(\Omega)$  such that

$$a(u, v) = Q(v) \quad \forall v \in H_0^1(\Omega) \quad (2.1)$$

where  $\Omega$  is a bounded region in  $R^2$  and

$$a(u, v) = \int_{\Omega} \left\{ \sum_{i,j=1}^2 a_{i,j}(x) \partial_i u \partial_j v + \sum_{i=1}^2 b_i(x) \partial_i u v + c(x) u v \right\} dx$$

$$Q(v) = \int_{\Omega} f v dx, \quad \forall v \in H_0^1(\Omega), x = (x_1, x_2) \in \Omega$$

and  $a_{i,j}, b_i, c \in L^\infty(\Omega)$ .

We assume that the bilinear form  $a(u, v)$  satisfies the condition: there exist positive constants  $\nu_1$  and  $\nu_2$  such that

$$\nu_1 \|u\|_{H_0^1(\Omega)}^2 \leq a(u, u), \quad \forall u \in H_0^1(\Omega) \quad (2.2)$$

and

$$a(u, v) \leq \nu_2 \|u\|_{H_0^1(\Omega)}^2 \cdot \|v\|_{H_0^1(\Omega)}, \quad \forall u, v \in H_0^1(\Omega)$$

This problem has an unique solution if the boundary  $\partial\Omega$  of the region  $\Omega$  is piecewise Lipschitz-continuous; see for example [3].

We solve the problem (2.1) by the finite element method with triangular elements and piecewise linear approximation. For simplicity, we assume that the boundary  $\partial\Omega$  of  $\Omega$  is a polygonal line, but a similar analysis applies for the general form of  $\partial\Omega$ . The region  $\Omega$  is partitioned into triangles  $e_i$  such that the intersection of two different triangles is either empty or consist of exactly one vertex of exactly one side. The element  $e_i$  is characterized by its greatest side  $h_i$  and we denote  $\max_i h_i$  by  $h$ .

For a given partition we define the finite element space

$$v_h(\Omega) = \{v \in C(\Omega) \mid v|_{e_i} \in P_1(x); v(x) = 0, \forall x \in \partial\Omega\}$$

Where  $P_1(x)$  is the set of linear polynomials. A function  $v$  on  $e_i$  is represented by its values at vertices (nodes) of  $e_i$ . The approximation problem for (2.1) in the space  $V_h(\Omega)$  is the following.

Find a function  $u_h \in V_h(\Omega)$  such that

$$a(u_h, v) = Q(v), \quad \forall v \in V_h(\Omega) \tag{2.3}$$

This problem has a unique solution. If the partition of  $\Omega$  is regular (see for example [3], p.132) and  $u \in H^2(\Omega) \cap H_0^1(\Omega)$ , then for  $s = 0, 1$

$$\|u - u_h\|_{H^s} \leq M h^{2-s} \|u\|_{H^2}$$

where  $u$  and  $u_h$  are the solution of the problems (2.1) and (2.3) respectively, and  $M$  is a positive constant independent of  $h$ .

Using the nodal basis of the space  $V_h(\Omega)$  we rewrite the problem (2.3) as the linear system

$$Au_h = f_h \tag{2.4}$$

where  $(Au_h, v_h)_{R^N} = a(u_h, v_h)$ ,  $(f_h, v_h)_{R^N} = Q(v_h)$  and  $N$  is the number of nodes belonging to  $\Omega$ . From now on we use notation that makes no distinction between the function  $v_h$  belonging to  $V_h(\Omega)$  and the grid function (vector)  $v_h$  if they are equal at the given nodes.

### 3. The Preconditioning Algorithm

For the solution of the linear system (2.4) we shall use the preconditioned conjugate gradient method (see [4] or [6]) of the form

$$\begin{aligned} Bu_h^{k+1} &= \alpha_{k+1}(B - \tau_{k+1}A^T B^{-1}A)u_h^k + (1 - \alpha_{k+1})Bu_h^{k-1} + \\ &\quad \alpha_{k+1}\tau_{k+1}A^T B^{-1}f \quad (k = 1, 2, \dots) \quad (3.1) \\ Bu_h^1 &= (B - \tau_1 A^T B^{-1}A)u_h^0 + \tau_1 A^T B^{-1}f, \end{aligned}$$

where  $u_0 \in R^N$  is a given vector and the parameters  $\alpha_{k+1}$  and  $\tau_{k+1}$  are equals to

$$\tau_{k+1} = \frac{(r_k B^{-1} r_k)_{R^N}}{(B^{-1} A B^{-1} r_k, A B^{-1} r_k)_{R^N}}, \quad (k = 0, 1, 2, \dots) \quad (3.2)$$

$$\alpha_{k+1} = \left\{ 1 - \frac{\tau_{k+1}}{\tau_k} \frac{(r_k, B^{-1} r_k)_{R^N}}{(r_{k-1}, B^{-1} r_{k-1})_{R^N}} \frac{1}{\alpha_k} \right\}^{-1},$$

and

$$r_k = A^T B^{-1}(AU_h^k - f).$$

In the above expression by  $B$  we denote a symmetric positive definit matrix. The construction of  $B$  we give in section 5. We

assume that the partition of  $\Omega$  is regular and the systems involving matrix  $B$  is solved by a direct method. Let  $w_1(B)$  denote the cost of the factorization of  $B$ , i.e.  $B = LU$ , by a direct method and  $w_2(B)$  - the cost of solving the system with  $B$  when the factorization  $B = LU$  is given.

On the convergence of iterative method (3.1), (3.2) we can prove the following theorem.

**Theorem 1.:** With the symmetric positive definite matrix  $B$  (given in section 5) the computational work to find solution  $u_h$  of (2.4) with the accuracy  $\epsilon$ , i.e.

$$\| u_h - u_h^k \|_{H^1(\Omega)} \leq \epsilon \tag{3.3}$$

requires of order

$$w_1(B) + w_2(B) \ln(\epsilon^{-1}) \tag{3.4}$$

arithmetic operations, where  $u_h^k$  is computed by the method (3.1) and (3.2) in  $k$  iterations.

#### 4. The Construction of the Preconditioner

In this point we give the construction of the preconditioner  $B$  from (3.1) in the form of bilinear form  $b(.,.)$ . The algorithm given here we use to build up a simple algorithm for solving the equation  $Bx = y$  described in the point 5.

Let  $\Gamma$  denote a curve contained in  $\Omega$  and consisting only the certain sides of the triangular elements. It divides the region  $\Omega$  into two subregions  $\Omega_1$  and  $\Omega_2$ . Suppose that the partition of  $\Omega$  is regular and the number of pieces of  $\Gamma$  is finite when  $h$  tends to zero (see figure 1.).

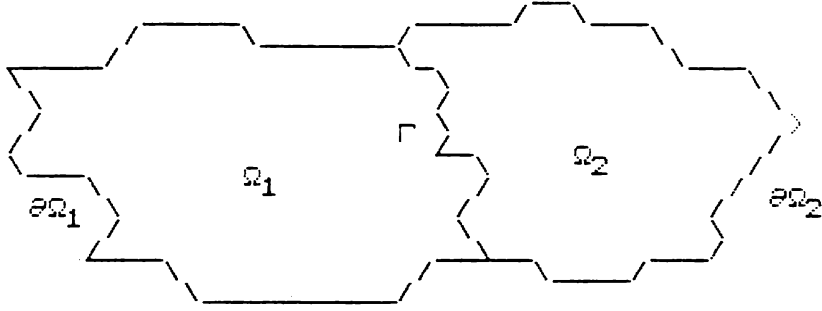


Figure 1.

In order to construct the preconditioner, we have to define three finite element spaces, related to  $V_h(\Omega)$ . Let  $V_h(\Omega_1)$  and  $V_h(\Omega_2)$  be the restriction of elements in  $V_h(\Omega)$  which vanish in  $\Omega_2$  and  $\Omega_1$  respectively and in particular on  $\Gamma$  and let  $V_h(\Gamma)$  consist of those element in  $V_h(\Omega)$  which vanish in  $\Omega/\Gamma$ . Let

$$\tilde{a}(u, v) = \int_{\Omega} \left\{ \sum_{i,j=1}^2 \tilde{a}_{i,j}(x) \partial_i u \partial_j v + \tilde{c}(x) uv \right\} dx$$

$$\forall u, v \in V_h(\Omega) \quad (4.1)$$

where  $\tilde{a}_{i,j}, \tilde{c} \in L^\infty(\Omega)$ , a suitably choiced selfadjoint uniform elliptic bilinear form. That is it satisfies the following conditions:

$$\tilde{a}(u, v) = \tilde{a}(v, u), \forall u, v \in V_h(\Omega) \quad (4.2)$$

$$\tilde{\nu}_1 \|u\|_{H_0^1(\Omega)}^2 \leq \tilde{a}(u, u), \forall u \in V_h(\Omega)$$

and

$$\tilde{a}(u, v) \leq \tilde{\nu}_2 \|u\|_{H_0^1(\Omega)} \cdot \|v\|_{H_0^1(\Omega)}, \forall u, v \in V_h(\Omega)$$

where  $\tilde{\nu}_1$  and  $\tilde{\nu}_2$  are positive constants.

Set

$$\tilde{a}(u, v) = \tilde{a}_1(u, v) + \tilde{a}_2(u, v), \forall u, v \in V_h(\Omega) \quad (4.3)$$

on  $V_h(\Omega)$ , where

$$\tilde{a}_k(u, v) = \int_{\Omega_k} \left\{ \sum_{i,j=1}^2 \tilde{a}_{i,j}(x) \partial_i u \partial_j v + \tilde{c}(x) uv \right\} dx$$

$$\forall u, v \in V_h(\Omega) \quad (k = 1, 2)$$

Let us now consider an arbitrary function  $w \in V_h(\Omega)$ . We decompose  $w$  on  $\Omega$ , as follows. Let  $w = w_p + w_H$  where  $w_p \in V_h(\Omega)$  and satisfies

$$\tilde{a}_2(w_p, v) = \tilde{a}_2(w, v) \quad \forall v \in V_h(\Omega_2)$$

Notice that  $w_p$  is determined on  $\Omega_2$  by the values of  $w$  on  $\Omega_2$  and

$$a_2(w_H, v) = 0 \quad \forall v \in V_h(\Omega_2)$$

Thus on  $\Omega_2$   $w$  is decomposed into a function  $w_H$  which satisfies the above homogeneous equations.

We now can define our preconditioner in the form of

$$b(u, v) = \tilde{a}_1(u, v) + \tilde{a}_2(u_p, v_p) \quad \forall u, v \in V_h(\Omega) \quad (4.4)$$

On the basis of (4.1) and (4.2) easy to prove, that this bilinear form is also selfadjoint and uniform elliptic.

We shall show that the equations

$$b(w, v) = Q_\sigma(v) \quad \forall v \in V_h(\Omega) \quad (4.5)$$

where

$$Q_g(v) - \int_{\Omega} g v dx \quad \forall v \in V_h(\Omega) \text{ and } g \in L^2(\Omega)$$

given in the above form we can solve by solving related Galjorkin equations on  $\Omega_1$  and  $\Omega_2$  separately.

The algorithm for the solution of (4.5) can be defined in the following four steps:

(i) Consider  $v \in V_h(\Omega_2)$ . Then (4.5) reduces to

$$\tilde{a}_2(w_p, v) = Q_g(v) \quad \forall v \in V_h(\Omega_2) \quad (4.6)$$

Since  $w_p \in V_h(\Omega_2)$  this is just the solution of a discrete Dirichlet problem on  $\Omega_2$ .

(ii) With  $w_p$  known, we can write (4.5) as

$$\begin{aligned} \tilde{a}_1(w, v) &= Q_g(v) - \tilde{a}_2(w_p, v_p) = \\ &= Q_g(v) - \tilde{a}_2(w_p, v_p) \quad \forall v \in V_h(\Omega) \end{aligned} \quad (4.7)$$

The last equality is true because  $\tilde{a}_2(w_p, v_H) = 0$ . The equations (4.7) uniquely determine  $w$  on  $\Omega_1$ . In fact  $w|_{\Omega_1} (\in V_h(\Omega_1))$  is the discrete solution of a mixed Neumann-Dirichlet problem on  $\Omega_1$ . So we have  $w$  on  $\Omega_1$  and, in particular on  $\Gamma$ .

(iii) We can now determine  $w_H$  on  $\Omega_2$  as the solution of a homogeneous Dirichlet problem with values  $w|_{\Gamma}$  on  $\Gamma$  and zero on the rest of  $\partial\Omega_2$ . That is by solution of the equation

$$\tilde{a}_2(w_H, v) = 0 \quad \forall v \in V_h(\Omega_2). \quad (4.8)$$

(iv) Let

$$w = w_p + w_H \quad (4.9)$$

## 5. The Block Matrix Representation of the Preconditioner



In this section we will describe the preconditioner  $B$  in the terms of block matrices. It will be shown that  $B$  - the matrix representation of the bilinear form  $b(.,.)$  - has a special structure and the process for solving (4.5) previously described may also be seen as a block Gauss elimination process. We shall suppose that we have the usual nodal basis for  $V_h(\Omega)$  and the nodes are partitioned into three subset correspondingly to those in  $\Gamma$ ,  $\Omega_1$  and  $\Omega_2$ . We order the vectors as follows

$$v = \begin{bmatrix} v_0 \\ v_1 \\ v_2 \end{bmatrix} \quad (5.1)$$

where  $v_0, v_1$  and  $v_2$  corresponding to the nodes on  $\Gamma, \Omega_1$  and  $\Omega_2$ , respectively. So the equations (4.5) can be rewrite into the following linear system

(5.2)

$$\begin{bmatrix} B_{00} + B_{02}B_{22}^{-1}B_{01} & B_{01} & B_{02} \\ B_{01}^T & B_{11} & 0 \\ B_{02}^T & 0 & B_{22} \end{bmatrix} \begin{bmatrix} w_0 \\ w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} Q_{g_0} \\ Q_{g_1} \\ Q_{g_2} \end{bmatrix}$$

where the submatrices and the  $Q_g$  vector are

$$\begin{aligned} \begin{bmatrix} B_{00} & B_{01} \\ B_{01}^T & B_{11} \end{bmatrix}_{i,j} &= \tilde{a}_1(\varphi_i, \varphi_j), & \varphi_i, \varphi_j \in V_h(\Omega_1) \\ (B_{22})_{i,j} &= \tilde{a}_2(\varphi_i, \varphi_j), & \varphi_i, \varphi_j \in V_h(\Omega_2) \\ (B_{02})_{i,j} &= \tilde{a}_1(\varphi_i, \varphi_j), & \varphi_i \in V_h(\Gamma), \varphi_j \in V_h(\Omega_2) \\ (Q_g)_i &= Q_g(\varphi_i), & \varphi_i \in V_h(\Omega) \end{aligned}$$

In the expressions by  $\varphi_i$  we denote the elements of the nodal basis  $V_h(\Omega)$ .

Using (4.6) - (4.9) the algorithm for the solution of (5.2) we can define in the following four steps.

(i) Solve the system

$$B_{22}(w_p)_2 = Q_{g_2} \quad (5.3)$$

This corresponds to the reformulation of the system (5.2) into

$$\begin{bmatrix} B_{00} & B_{01} & 0 \\ B_{01}^T & B_{11} & 0 \\ B_{02}^T & 0 & B_{22} \end{bmatrix} \begin{bmatrix} w_0 \\ w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} Q_{g_0} - B_{02}B_{22}^{-1}Q_{g_2} \\ Q_{g_1} \\ Q_{g_2} \end{bmatrix} \quad (5.4)$$

(ii) Solve the system

$$\begin{bmatrix} B_{00} & B_{01} \\ B_{01} & B_{11} \end{bmatrix} \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} = \begin{bmatrix} Q_{g_0} - B_{02}B_{22}^{-1}Q_{g_2} \\ Q_{g_1} \end{bmatrix} \quad (5.5)$$

(iii) Solve the system

$$B_{22}(w_H)_2 = -B_{02}^T w_0 \quad (5.6)$$

because  $w_0$  and  $w_1$  are known.

(iv) Let

$$w = w_H + w_p. \quad (5.7)$$

## 6. Proof of Theorem 1

To prove the Theorem 1 from section 3. we need a few auxiliary lemmas. Let  $N$  denote the number of the nodes belonging to  $\Omega$ .

**Lemma 1:** Let (2.2) satisfies and suppose that the partition of  $\Omega$  is regular and the number of pieces of  $\Gamma$  (see section 4) is finite when  $h$  tends to zero. Then

$$\begin{aligned} \delta_1 (Bv, v)_{R^N} \leq (Av, v)_{R^N} \leq \delta_2 (Bv, v)_{R^N} \\ \forall v \in R^N \end{aligned} \quad (6.1)$$

where  $\delta_1$  and  $\delta_2$  are positive constants independent of  $h$ , and  $A$  and  $B$  matrices from (2.3) and (5.2), respectively.

**Proof:** By definition of  $A$  and  $B$  easy to see that (6.1) is equivalent with the

$$\delta_1 b(v, v) \leq a(v, v) \leq \delta_2 b(v, v), \quad \forall v \in V_h(\Omega) \quad (6.2)$$

inequalities. From the definition of  $\tilde{a}(\cdot, \cdot)$  it follows that

$$c_1 \tilde{a}(v, v) \leq a(v, v) \leq c_2 \tilde{a}(v, v), \quad \forall v \in V_h(\Omega)$$

where  $c_1$  and  $c_2$  are positive constants independent of  $h$ .

Since

$$c_1 b(v, v) \leq c_1 \tilde{a}(v, v) \leq a(v, v), \quad \forall v \in V_h(\Omega) \quad (6.3)$$

we can choose  $\delta_1 = c_1$ . Thus we need to prove that

$$a(v, v) \leq \delta_2 b(v, v), \quad \forall v \in V_h(\Omega)$$

i.e.

$$c_2 \tilde{a}(v, v) \leq \delta_2 b(v, v), \quad \forall v \in V_h(\Omega) \quad (6.4)$$

which follows from the inequality

$$\tilde{a}_2(v_H, v_H) \leq c_3 \tilde{a}_1(v, v), \quad \forall v \in V_h(\Omega) \quad (6.5)$$

where  $c_3$  independent of  $h$ . The inequality (6.5) is proved as follows: Let  $w$  an arbitrary function from  $V_h(\Omega)$ . Then the restriction of  $w_H$  to  $\Omega_2$  by definition is the solution of the homogeneous discrete Dirichlet problem

$$w_H = w \text{ on } \Gamma$$

and

$$\tilde{a}_2(w_H, v) = 0, \quad \forall v \in V_h(\Omega_2)$$

From this we have that

$$\begin{aligned} \tilde{a}_2(w_H, w_H) &\leq \tilde{a}_2(v, v), \\ \forall v \in \{u \in V_h(\Omega) : u|_\Gamma = w|_\Gamma\} \end{aligned} \quad (6.6)$$

Since

$$\begin{aligned} \tilde{a}_2(v, v) &= \\ &= \tilde{a}_2(v - w_H + w_H, v - w_H + w_H) = \\ &= \tilde{a}_2(v - w_H, v - w_H) + 2\tilde{a}_2(w_H, w_H - v) + \tilde{a}_2(w_H, w_H) \geq \\ &\geq \tilde{a}_2(w_H, w_H) \end{aligned}$$

where we used that  $\tilde{a}_2(\cdot, \cdot)$  is a selfadjoint form and  $v - w_H \in V_h(\Omega_2)$ . Let choice  $\tilde{w}$  as a finite element extension of  $w|_{\Omega_1}$  to  $\Omega$  (see [7] Theorem 1). In this case the following estimation must hold

$$\|\tilde{w}\|_{H_0^1(\Omega)} \leq c_4 \|w\|_{H^1(\Omega_1)}$$

where  $c_4$  is a constant independent of  $h$  and  $w$ .

So using (6.6) and the ellipticity of  $\tilde{a}(\cdot, \cdot)$ , we get

$$\begin{aligned} \tilde{a}_2(w_H, w_H) &\leq \tilde{a}_2(\tilde{w}, \tilde{w}) \leq \tilde{\nu}_2 \|\tilde{w}\|_{H^1(\Omega_2)}^2 \leq \\ &\leq \tilde{\nu}_2 c_4 \|\tilde{w}\|_{H^1(\Omega_1)}^2 = \\ &= \tilde{\nu}_2 c_4 \|w\|_{H^1(\Omega_1)}^2 \leq \tilde{\nu}_1 \tilde{\nu}_2 c_4 \tilde{a}_1(w, w) \end{aligned}$$

From this the inequality (6.5) follows with  $c_3 = \tilde{\nu}_1 \tilde{\nu}_2 c_4$  and with  $\delta_2 = c_2(1 + c_3)$  the inequality (6.4) satisfies too.

**Lemma 2:** Let the assumption of Lemma 1 is satisfied. Then

$$\begin{aligned} \delta_3 (Bv, v)_{R^N} &\leq (B^{-1}Av, Av)_{R^N} \leq \delta_4 (Bv, v)_{R^N}, \\ \forall v &\in R^N \end{aligned} \tag{6.7}$$

where  $\delta_3$  and  $\delta_4$  are positive constants independent of h.

**Proof:** (i) The first inequality of (6.7) we can prove using the Cauchy - Schwarz inequality in the following way

$$\begin{aligned} 0 &\leq (Bv, v)_{R^N} \leq \frac{1}{\delta_1} (Av, v)_{R^N} \leq \frac{1}{\delta_1} (B^{1/2}Av, B^{1/2}v)_{R^N} \leq \\ &\leq \frac{1}{\delta_1} (B^{1/2}Av, B^{1/2}Av)_{R^N}^{1/2} \cdot (B^{1/2}v, B^{1/2}v, B^{1/2}v)_{R^N}^{1/2}, \forall v \in R^N \end{aligned}$$

i.e

$$(Bv, v)_{R^N} \leq \frac{1}{\delta_1} 2(B^{-1}Av, Av)_{R^N}$$

From this we have that with  $\delta_3 = \delta_1^2$  first inequality of (6.7) holds.

(ii) The B matrix is symmetric and positive definit thus

$$(B^{-1}Av, Av)_{R^N} = \| B^{-1/2}Av \|_{R^N}^2, \quad \forall v \in R^N$$

and

$$\begin{aligned} \| B^{-1/2}Av \|_{R^N} &= \max_{z \in R^N} \frac{(B^{-1/2}Av, z)_{R^N}}{\| z \|_{R^N}} = \\ &= \max_{g \in R^N} \frac{(Av, g)_{R^N}}{\| B^{1/2}g \|_{R^N}} \end{aligned}$$

where  $g = B^{-1/2}z$ . Using the Lemma 1 and the fact that the bilinear form  $a(.,.)$  is uniform elliptic we have

$$\begin{aligned} \max_{g \in R^N} \frac{(Av, g)_{R^N}}{\|B^{1/2}g\|_{R^N}} &\leq \max_{g \in R^N} \frac{\nu_2 \|v\|_{H_0^1(\Omega)} \cdot \|g\|_{H_0^1(\Omega)}}{\|B^{1/2}g\|_{R^N}} \leq \\ \nu_1 \nu_2 \max_{g \in R^N} \frac{(Av, v)_{R^N}^{1/2} \cdot (Ag, g)_{R^N}^{1/2}}{\|B^{1/2}g\|_{R^N}} &\leq \nu_1 \nu_2 \delta_2^{1/2} \cdot (Av, v)_{R^N}^{1/2} \\ &\leq \nu_1 \nu_2 \delta_2 \cdot (Bv, v)_{R^N}^{1/2} \end{aligned}$$

From this with  $\delta_4 := \nu_1^2 \nu_2^2 \delta_2^2$  we get the second inequality of (6.7).

**Proof (of Theorem 1):** The convergence of method (3.1), (3.2) and satisfaction of (3.4) follow from the uniform ellipticity of bilinear form  $\tilde{a}(.,.)$  and the relations (6.7) (see [4], [6]).

**Remark:** Our method can be generalized easily for the higher order elliptic operators and for the higher order finite elements too (see [7]).

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