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ON A GENERAL STACKELBERG-TYPE LEADER-FOLLOWER OLIGOPOLY MODEL

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Abstract. This paper introduces a new leader-follow type N-persons game, in which the players are divided into K groups. It is assumed that group k follows group $k-1$ for $k>1$, and group 1 follows group K. The players belonging to a group form equilibrium among each other under the newest informations available. The formulation of the model is given and in the linear case stability conditions are derived.

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1. Introduction

The stability of the Cournot-Nash equilibrium has been studied by many authors under different assumptions on the behavior of the firms. The classical oligopoly game was first discussed under the expectations a la Cournot by Theocharis (1960). His result has been extended to multiproduct oligopoly by Szidarovszky and Okuguchi (1986). Models with adaptive expectation have been analyzed by Fisher (1961) and Okuguchi (1970, 1976). These classical results given for the classical ologopoly game have been generalized to multiproduct models by Okuguchi and Szidarovszky (1987a). The stability of the Stackelberg duopoly and that of a Stackelberg-type oligopoly have been analyzed by Hathaway, Howroyd and Rickard (1979), and by Okuguchi (1979), respectively, and a multiple leader Stackelberg model has been introduced and analyzed by Sherali (1984). A sequential adjustment process has been introduced by Gobay and Moulin (1980), and its generalization to multiproduct oligopoly has been presented in Okuguchi and Szidarovszky (1987b).

In this paper a generalized version of this sequential adjustment process will be introduced. We formulate the dynamic process as follows: Assume that the players are divided into disjoint groups $G_1, G_2, \ldots, G_K$. At time $t = 0$, each player selects and initial strategy, and for each time $t > 0$ the following sequential adjustment process is performed. First, group G_1 forms equilibrium under the latest information $\underline{x}_j^{(t-1)}$ $(j \in G_k, k \neq 1)$ available. Then, group G_2 forms equilibrium under fixed strategies $\underline{x}_j^{(t)}$ $(j \in G_1)$ and $\underline{x}_j^{(t-1)}$ $(j \in G_k, k > 2)$, and so on. Finally group G_K forms equilibrium with fixed strategies $\underline{x}_j^{(t)}$ $(j \in G_k, k \leq K-1)$. If each group contains only one player, then this process coincides with the original sequential adjustment process of Gabay and Moulin (1980). In the further special case where $K = 2$ this model reduces to the leader-follower duopoly model of Stackelberg (1934).

The development of the paper is the following. First, the mathematical model will be formulated. Then, stability conditions will be presented on the basis of the stability theory of linear difference equations.

2. The Mathematical Formulation of the Model

Assume that in N-person game $X_i (i = 1, 2, \ldots, N)$ is the set of strategies of player i, and ϕ_i is his payoff function. Then this game is denoted by $\Gamma = \{N, X_1, \ldots, X_N, \phi_1, \ldots, \phi_N\}$.

Assume furthermore that the players are divided into disjoint groups $G_1, G_2, \ldots, G_K$, such that $G_k \cap G_\ell = \phi$ if $k \neq \ell$, and

$G_1 \cup G_2 \cup \ldots \cup G_K = \{1, 2, \ldots, N\}$. Let $x_i^{(0)}$ denote the strategy of player $i (1 \leq i \leq N)$ at $t = 0$. Each time period $t > 0$ and $k (1 \leq k \leq K)$ an equilibrium is formed by the players of group G_k with fixed values of $\underline{x}_i = \underline{x}_i^{(t)} (i \in G_1, 1 < k)$ and $\underline{x}_i = \underline{x}_i^{(t-1)} (i \in G_1, 1 > k)$. Assume the uniqueness of this equilibrium and denote it by

$$\underline{x}^{(k)} = \underline{E}^{(k)} \left(\underline{x}^{(1)(t)}, \ldots, \underline{x}^{(k-1)(t)}, \underline{x}^{(k+1)(t-1)}, \ldots, \underline{x}^{(K)(t-1)} \right), \tag{1}$$

where

$$\underline{x}^{(1)} = (\underline{x}_j)_{j \in G_1}.$$

Then $\underline{x}^{(k)(t)} = \underline{x}^{(k)}$, and therefore this general sequential adjustment process can be described by the Gauss-Seidel type (see Szidarovszky and Yakowitz, 1978) iteration

$$\underline{x}^{(k)(t)} = E^{(k)} \left(\underline{x}^{(1)(t)}, \ldots, \underline{x}^{(k-1)(t)}, \underline{x}^{(k+1)(t-1)}, \ldots, \underline{x}^{(K)(t-1)} \right),$$

$$(k = 1, \ldots, K) \tag{2}$$

Definition. The Nash equilibrium $u^* = \left(\underline{x}^{(1)*}, \ldots, \underline{x}^{(K)*} \right)$ is said to be stable under the above general sequential adjustment process if starting from arbitrary initial strategies $x_i^{(0)} (1 \leq i \leq N)$, the process (2) converges to $\underline{x}^*$.

For $k = 1, 2, \ldots, K$ define $X^{(k)} = \times_{j \in G_k} X_j$, and assume that m_k is the dimension of $X^{(k)}$. Let $\| \cdot \|_k$ denote a vector norm in $R^{m_k}, k = 1, 2, \ldots, K$.

Assume that

a) All sets $X_i (1 \leq i \leq N)$ are closed in R^{n_i};

b) There exist nonnegative real number $a_{kl} (k, l = 1, \ldots, K; l \neq k)$ such that for all k,

$$\sum_{l=1, l \neq k}^{K} a_{kl} < 1, \tag{3}$$

and for arbitrary $\underline{x}^{(\ell)}, \underline{y}^{(\ell)} \in X^{(\ell)}$,

$$\begin{aligned} &\| E^{(k)}\left(\underline{x}^{(1)}, \ldots, \underline{x}^{(k-1)}, \underline{x}^{(k+1)}, \ldots, \underline{x}^{(K)}\right) - \\ &- E^{(k)}\left(\underline{y}^{(1)}, \ldots, \underline{y}^{(k-1)}, \underline{y}^{(k+1)}, \ldots, \underline{y}^{(K)}\right) \|_k \\ &\leq \sum_{l=1, l \neq k}^{K} a_{kl} \| \underline{x}^{(\ell)} - \underline{y}^{(\ell)} \|_\ell . \end{aligned} \tag{4}$$

The following theorem is well known. See Szidarovszky and Yakowitz (1978).

Theorem 1. Under assumption a) and b) the Nash-equilibrium is stable under the dynamic process (2).

Since the condition of the theorem are only seldom satisfied, this result does not have a broad application in economics. More practical condition can be obtained for special games, when mappings $e^{(k)}$ can be determined in closed form, and they have special structures. In the next section of this paper such a special case will be introduced and investigated.

3. Stability Conditions for a Linear Multi-Product Oligopoly Model

The classical oligopoly game has been introduced by Cournot (1838), and his model has been generalized and analyzed by many authors. Okuguchi (1976) and Friedman (1979, 1981, 1986) give a survey of related works. The multiproduct oligopoly game was introduced by Selten (1970) and Szidarovszky (1978) for quantity strategies and by Eichhorn (1971a, b) for price strategies. The model of Szidarovszky (1978) will be now examined.

Assume a market with N firms and assume also that each of them produces M kinds of products. If $x_k^{(m)} (1 \leq k \leq N, 1 \leq k \leq M)$ denotes the production level of firm k of product m, then the output of firm k is characterized by an output vector $\underline{x}_k = (x_k^{(1)}, \ldots, x_k^{(M)})$. This vector $\underline{x}_k$ is considered to be the strategy of firm (player) k. Let the production cost of firm k be denoted by $C_k(\underline{x}_k)$, and assume that the unit price P_m of product m depends on the total output vector

$$\underline{s} = \left(\sum_{k=1}^{N} x_k^{(1)}, \ldots, \sum_{k=1}^{N} x_k^{(M)}\right)$$

of the industry. Using these notations the payoff function (or profit) of firm k can be formulated as

$$\phi_k(x_1, \ldots, \underline{x}_N) = \underline{x}_k^T \underline{p}(s) - C_k(\underline{x}_k), \tag{5}$$

where $\underline{p} = (p_1, \ldots, P_M)^T$. The following assumption are made:

(A) The feasible output set X_k (or set of strategies) of firm $k (1 \leq k \leq N)$ is a closed, convex, bounded set in R^M such that $\underline{x}_k \in X_k$ and $0 \leq \underline{t}_k \leq \underline{x}_k$ implies that $\underline{t}_k \in X_k$;

(B) For all $\underline{s}$,

$$\underline{p}(\underline{s}) = \underline{\underline{A}}\underline{s} + \underline{b}, \tag{6}$$

where $\underline{\underline{A}}$ and $\underline{b}$ are constant matrix and vector, respectively; furthermore matrix $\underline{\underline{A}} + \underline{\underline{A}}^T$ is negative definite;

(C) For all k,

$$C_k(\underline{x}_k) = \underline{b}_k^T \underline{x}_k + c_k, \tag{7}$$

where $\underline{b}_k > 0$ is a constant vector, and c_k is a scalar.

It is known (Szidarovszky and Okuguchi, 1987), that under conditions (A), (B) and (C) there exists at least one equilibrium point, which is called the Cournot-Nash equilibrium of the multi-product oligopoly game.

Under the above assumption at each time subperiod $t(k)$, each firm i from group $G_k (k = 1,\dots,K)$ maximizes his profit with fixed output vector $\underline{x}_j = \underline{x}_j^{(t)}$, if $j \in G_\ell(\ell < k$ or $\ell = k$ with $j \neq i)$ and with $\underline{x}_j = \underline{x}_j^{(t-1)}$, if $j \in G_\ell(1 > k)$. Assuming that the resulted output of firm i in the equilibrium is an interior point of X_i, the first order optimality conditions imply that for all $i(1 \leq i \leq N)$,

$$(\underline{\underline{A}} + \underline{\underline{A}}^T)\underline{x}_i^{(t)} + \underline{\underline{A}}\{\sum_{\ell<k}\sum_{j\in G_\ell} \underline{x}_j^{(t)} + \sum_{j\in G_k, j\neq i} \underline{x}_j^{(t)} + \sum_{\ell>k}\sum_{j\in G_\ell} \underline{x}_j^{(t-1)}\} +$$

$$+\underline{b} - \underline{b}_i = 0. \qquad (i \in G_k)$$

These equalities are equivalent to the relations

$$\underline{\underline{H}}_{kk}\underline{x}^{(k)(t)} + \sum_{\ell<k} \underline{\underline{H}}_{k\ell}\underline{x}^{(\ell)(t)} + \sum_{\ell>k} \underline{\underline{H}}_{k\ell}\underline{x}^{(\ell)(t-1)} + \underline{\alpha}_k = \underline{0}$$

$$(k = 1,2,\dots,K) \tag{8}$$

where for all $\ell, \underline{x}^{(1)} = (\underline{x}_i)_{i\in G_1}, \underline{\alpha}_k$ is a constant vector,

$$\underline{\underline{H}}_{kk} = \begin{pmatrix} \underline{\underline{A}} + \underline{\underline{A}}^T & \underline{\underline{A}} & \dots & \underline{\underline{A}} \\ \underline{\underline{A}} & \underline{\underline{A}} + \underline{\underline{A}}^T & \dots & \underline{\underline{A}} \\ \vdots & \vdots & & \vdots \\ \underline{\underline{A}} & \underline{\underline{A}} & \dots & \underline{\underline{A}} + \underline{\underline{A}}^T \end{pmatrix}, \tag{9}$$

and

$$\underline{\underline{H}}_{kl} = \begin{pmatrix} \underline{\underline{A}} & \dots & \underline{\underline{A}} \\ \vdots & & \vdots \\ \underline{\underline{A}} & \dots & \underline{\underline{A}} \end{pmatrix} \qquad (\forall k, l) \tag{10}$$

Note that the types of matrices $\underline{\underline{H}}_{kk}$ and $\underline{\underline{H}}_{jl}$ are $(|G_k|\cdot M)\times$ $(|G_k|\cdot M)$ and $(|G_k|\cdot M)\times(|G_\ell|\cdot M)$, respectively, where for all k, $|G_k|$ denotes the number of firms from group G_k.

Our main result is the following

Theorem 2. Assume that $\underline{\underline{A}} = \underline{\underline{A}}^T$, and conditions (A), (B) and (C) hold. Then the equilibrium is stable under the dynamic process (8).

Proof. The proof of the theorem consists of severel stages.

a) First we prove that matrix

$$\underline{\underline{H}} = \begin{pmatrix} \underline{\underline{H}}_{11} & \underline{\underline{H}}_{12} & \cdots & \underline{\underline{H}}_{1K} \\ \underline{\underline{H}}_{21} & \underline{\underline{H}}_{22} & \cdots & \underline{\underline{H}}_{2K} \\ \vdots & \vdots & & \vdots \\ \underline{\underline{H}}_{K1} & \underline{\underline{H}}_{K2} & \cdots & \underline{\underline{H}}_{KK} \end{pmatrix} \tag{11}$$

is negative definite. Relations (9) and (10) imply that $\underline{\underline{H}}$ is the Kronecker product of matrices $\underline{\underline{A}}$ and $\underline{\underline{1}}+\underline{\underline{I}}$, where $\underline{\underline{1}}$ is the matrix, the elements of which are all equal to unity. Since $\underline{\underline{A}}$ is symmetric, $\underline{\underline{H}}$ is symmetric, and its eigenvalues can be obtained as the products of the eigenvalues of $\underline{\underline{A}}$ and $\underline{\underline{1}}+\underline{\underline{I}}$. Since $\underline{\underline{A}}+\underline{\underline{A}}^T = 2\underline{\underline{A}}$ is negative definite, the eigenvalues of $\underline{\underline{A}}$ are all negative. One may easily verify that the eigenvalues of $\underline{\underline{1}}+\underline{\underline{I}}$ are equal either to 1 or $N+1$. Hence all eigenvalues of $\underline{\underline{H}}$ are negative.

b) Introduce next the notations

$$\underline{\underline{E}} = \begin{pmatrix} \underline{\underline{H}}_{11} & & & \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ & & & & \underline{\underline{H}}_{KK} \end{pmatrix},$$

$$\underline{\underline{L}} = \begin{pmatrix} \underline{\underline{0}} & & & & 0 \\ \underline{\underline{H}}_{21} & \underline{\underline{0}} & & & \\ \vdots & \vdots & & & \\ \underline{\underline{H}}_{K1} & \underline{\underline{H}}_{K2} & \cdots & \underline{\underline{H}}_{K,K-1} & \underline{\underline{0}} \end{pmatrix}$$

and

$$\underline{\underline{U}} = \begin{pmatrix} \underline{\underline{0}} & \underline{\underline{H}}_{12} & \cdots & \underline{\underline{H}}_{1K} \\ & \underline{\underline{0}} & \cdots & \underline{\underline{H}}_{2K} \\ & & & \vdots \\ & & \underline{\underline{0}} & \underline{\underline{H}}_{K-1,K} \\ & 0 & & \\ & & & \underline{\underline{0}} \end{pmatrix}, \tag{12}$$

then equation(8) implies that

$$\underline{x}^{(k)(t)} = -\sum_{\ell<k} \underline{\underline{H}}_{kk}^{-1} \underline{\underline{H}}_{k\ell} \underline{x}^{(\ell)(t)} - \sum_{\ell>k} \underline{\underline{H}}_{kk}^{-1} H_{k\ell} \underline{x}^{(\ell)(t-1)} + \underline{\beta}_k,$$

where $\underline{\beta}_k$ is a constant vector. By using notation (12) we can rewrite this equation as

$$\underline{x}^{(t)} = -(\underline{\underline{I}} + \underline{\underline{E}}^{-1}\underline{\underline{L}})^{-1}(\underline{\underline{E}}^{-1}\underline{\underline{U}})\underline{x}^{(t-1)} + \underline{\beta},$$

where $\underline{\beta}$ is a constant vector, and $\underline{x}^{(t)} = \left(\underline{x}^{(k)(t)}\right)_{k=1}^{K}$. We shall next prove that the eigenvalues of matrix

$$\underline{\underline{Q}} = -(\underline{\underline{I}} + \underline{\underline{E}}^{-1}\underline{\underline{L}})^{-1}\underline{\underline{E}}^{-1}\underline{\underline{U}}$$

are all inside the unit disk of the complex plane, which implies the assertion of the theorem.

Simple calculation shows that $\underline{\underline{Q}}$ can be rewritten as

$$\underline{\underline{Q}} = -(\underline{\underline{E}} + \underline{\underline{L}})^{-1}\underline{\underline{U}}.$$

Consider now the eigenvalue problem of matrix $\underline{\underline{Q}}$,

$$-(\underline{\underline{E}} + \underline{\underline{L}})^{-1}\underline{\underline{U}}\underline{u} = \lambda\underline{u},$$

that is,

$$-\underline{\underline{U}}\underline{u} = \lambda(\underline{\underline{E}} + \underline{\underline{L}})\underline{u}. \tag{13}$$

P premultiplying both sides by $\underline{u}^*$ (which is the conjugate transposed of $\underline{u}$) we get

$$-z_1 + iz_2 = \lambda(\alpha + z_1 + iz_2), \tag{14}$$

where

$$\alpha = \underline{u}^*\underline{\underline{E}}\underline{u}, z_1 + iz_2 = \underline{u}^*\underline{\underline{L}}\underline{u},$$

furthermore with overbar denoting complex conjugate,

$$\underline{u}^*\underline{\underline{U}}\underline{u} = \underline{u}^*\underline{\underline{L}}^T\underline{u} = \overline{(\underline{u}^*\underline{\underline{U}}\underline{u})} = \overline{z_1 + iz_2} = z_1 - iz_2.$$

From (14) we conclude that

$$\lambda = \frac{-z_1 + iz_2}{\alpha + z_1 + iz_2}. \tag{15}$$

From the following argument it will also follow that the denominator differs from zero. We shall next prove that

$$| -z_1 + iz_2 | < | \alpha + z_1 + iz_2 | .$$

This inequality is equivalent to the relation

$$z_1^2 + z_2^2 < (\alpha + z_1)^2 + z_2^2,$$

that is

$$0 < \alpha^2 + 2\alpha z_1 = (\alpha + 2z_1)\alpha. \tag{16}$$

Hence we see that (16) implies the assertion. To prove that (16) holds consider the following inequalities:

$$0 > \underline{u}^* \underline{\underline{H}} \underline{u} = \underline{u}^* (\underline{\underline{L}} + \underline{\underline{U}} + \underline{\underline{E}}) \underline{u} =$$

$$= z_1 + iz_2 + z_1 - iz_2 + \alpha = \alpha + 2z_1, \tag{17}$$

$$0 > \underline{u}^* \underline{\underline{E}} \underline{u} = \alpha. \tag{18}$$

The first one is a simple consequence of the definition of α, z_1 and z_2. The second inequality is implied by the fact that all diagonal blocks of $\underline{\underline{H}}$ are necessarily negative definite. Thus (16) holds, which completes the proof.

4. A Generalized Process

In this section we shall generalize the process of the previous section.

Process (2) is the special case of the following more general iteration scheme. Assume that at each time subperiod $t(k)$, firms of group k determine their equilibrium

$$\underline{x}^{(k)E(t)} = E^{(k)}\left(\underline{x}^{(1)(t)}, \ldots, \underline{x}^{(k-1)(t)}, \underline{x}^{(k+1)(t-1)}, \ldots, \underline{x}^{(K)(t-1)}\right),$$

but they select vector

$$\underline{x}^{(k)(t)} = \underline{x}^{(k)(t-1)} + \underline{\underline{D}}_k \left(\underline{x}^{(k)E(t)} - \underline{x}^{(k)(t-1)}\right). \qquad (19)$$

Note that this process is a block variant of the successive overrelaxation (SOR) method for solving linear equation

$$\underline{\underline{H}}\underline{x} + \underline{\alpha} = \underline{0},$$

where $\underline{\alpha} = (\underline{\alpha}_k)_{k=1}^{K}$, and $\underline{\underline{H}}$ is defined in (11). This method is analyzed in great details by Ortega and Rheinboldt (1970). Matrix $\underline{\underline{D}}_k$ can be considered as the speed of adjustment in $\underline{x}^{(k)}$. In this paper we do not assume that matrices $\underline{\underline{D}}_k$ are diagonal.

We shall next prove a generalization of Theorem 2.

Theorem 3. Assume that $\underline{\underline{A}} = \underline{\underline{A}}^T$, and condition (A),(B) and (C) hold, furthermore matrices

$$\underline{\underline{D}}_k^T \underline{\underline{H}}_{kk} + \underline{\underline{H}}_{kk} \underline{\underline{D}}_k - \underline{\underline{D}}_k^T \underline{\underline{H}}_{kk} \underline{\underline{D}}_k$$

are negative definite for all k. Then the equilibrium is stable under the dynamic process (19).

Proof. From the proof of Theorem 2 we know that matrix $\underline{\underline{H}}$ is negative definite. Simple calculations show that process (19) can be rewritten as

$$\underline{x}^{(t)} = (\underline{\underline{I}} + \underline{\underline{D}}\underline{\underline{E}}^{-1}\underline{\underline{L}})^{-1}(\underline{\underline{I}} - \underline{\underline{D}} - \underline{\underline{D}}\underline{\underline{E}}^{-1}\underline{\underline{U}})\underline{x}^{(t-1)} + \underline{\gamma},$$

where $\underline{\gamma}$ is a constant vector, and $\underline{\underline{D}} = \text{diag}\,(\underline{\underline{D}}_1, \ldots, \underline{\underline{D}}_K)$. The matrix of coefficients of this difference equation can be written in the form

$$\underline{\underline{Q}}_1 = (\underline{\underline{I}} + \underline{\underline{E}}\underline{\underline{D}}^{-1}\underline{\underline{L}})^{-1}(\underline{\underline{I}} - \underline{\underline{D}} - \underline{\underline{D}}\underline{\underline{E}}^{-1}\underline{\underline{U}}) =$$

$$= (\underline{\underline{E}}\underline{\underline{\Delta}} + \underline{\underline{L}})^{-1}(\underline{\underline{E}}\underline{\underline{\Delta}} - \underline{\underline{E}} - \underline{\underline{U}}),$$

where $\underline{\underline{\Delta}} = \underline{\underline{D}}^{-1}$. We shall verify again that all eigenvalues of $\underline{\underline{Q}}_1$ are inside the unit disk of the complex plane. To prove this assertion consider the eigenvalue problem of matrix $\underline{\underline{Q}}_1$, which has th form:

$$(\underline{\underline{E}}\underline{\underline{\Delta}} - \underline{\underline{E}} - \underline{\underline{U}})\underline{u} = \lambda(\underline{\underline{E}}\underline{\underline{\Delta}} + \underline{\underline{L}})\underline{u}.$$

By premultiplying both sides by $\underline{u}^*$ we obtain the equation

$$w_1 + iw_2 - \alpha - z_1 + iz_2 = \lambda(w_1 + iw_2 + z_1 + iz_2),$$

where

$$\alpha = \underline{u}^*\underline{\underline{E}}\underline{u}, w_1 + iw_2 = \underline{u}^*\underline{\underline{E}}\underline{\underline{\Delta}}\underline{u}, z_1 + iz_2 = \underline{u}^*\underline{\underline{L}}\underline{u},$$

which implies that

$$\lambda = \frac{w_1 + iw_2 - \alpha - z_1 + iz_2}{w_1 + iw_2 + z_1 + iz_2}. \tag{20}$$

We shall finally verify that

$$| w_1 + iw_2 - \alpha - z_1 + iz_2 | < | w_1 + iw_2 + z_1 + iz_2 |,$$

which is equivalent to relation

$$(w_1 - \alpha - z_1)^2 + (w_2 + z_2)^2 < (w_1 + z_1)^2 + (w_2 + z_2)^2.$$

This can be simplified as

$$(\alpha - 2w_1)(\alpha + 2z_1) < 0. \tag{21}$$

Since $\underline{\underline{H}}$ is negative definite,

$$0 > \underline{u}^* (\underline{\underline{E}} + \underline{\underline{L}} + \underline{\underline{U}})\underline{u} = \alpha + z_1 + iz_2 + z_1 - iz_2 = \alpha + 2z_1,$$

furthermore

$$2w_1 - \alpha = w_1 + iw_2 + w_1 - iw_2 - \alpha = \underline{u}^* (\underline{\underline{E\Delta}} + \underline{\underline{\Delta}}^T \underline{\underline{E}} - \underline{\underline{E}})\underline{u} =$$

$$= \underline{v}^* (\underline{\underline{D}}^T \underline{\underline{E}} + \underline{\underline{ED}} - \underline{\underline{D}}^T \underline{\underline{ED}})\underline{v},$$

where $\underline{v} = \underline{\underline{D}}^{-1}\underline{u}$. Since matrices $\underline{\underline{D}}$ and $\underline{\underline{E}}$ are both block diagonal, the same is true for $\underline{\underline{D}}^T \underline{\underline{E}} + \underline{\underline{ED}} - \underline{\underline{D}}^T \underline{\underline{ED}}$ with diagonal block $\underline{\underline{D}}_k^T \underline{\underline{H}}_{kk} + \underline{\underline{H}}_{kk}\underline{\underline{D}} - \underline{\underline{D}}_k^T \underline{\underline{H}}_{kk}\underline{\underline{D}}_k$. Thus the conditions of the theorem imply that $2w_1 - \alpha < 0$. Thus (21) holds, which complete the proof of the theorem.

Remark 1. Theorem 1 can be obtained as a special case by selecting $\underline{\underline{D}}_k = \underline{\underline{I}}$ for all k.

Remark 2. Consider the special case when $\underline{\underline{D}}_k = w_k \underline{\underline{I}}$, where w_k is a constant number for all k. Then

$$\underline{\underline{D}}_k^T \underline{\underline{H}}_{kk} \underline{\underline{D}}_k - \underline{\underline{D}}_k^T \underline{\underline{H}}_{kk} \underline{\underline{D}}_k = (2w_k - w_k^2)\underline{\underline{H}}_{kk}.$$

Since $\underline{\underline{H}}_{kk}$ is negative definite, this matrix is negative definite if and only if $0 < w_k < 2$. This is the usual condition for the convergenc of the scalar SOR method.

Remark 3. If for all k, $| G_k | = 1$, then the sequential adjustment process discussed in Okuguchi and Szidarovszky (1987b) is obtained as a special case. Hence the results of this paper generalize the theorems of this earlier paper.

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