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ON SYSTEMS WITH BULK ARRIVAL AND GROUP SERVICE II.

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1. In [1] we investigated a queueing system with bulk arrival and group services described by means of an inhomogeneous Markov chain which can be used to determine the characteristics of operating systems. In the present paper we investigate similar problem in case when the local characteristics do not depend on time, we get an expression for the generating function of transition probabilites and derive the conditions of existence of ergodic distribution. The expression obtained for the generating function contains certain probabilities from the desired distribution, they can be determined as the solution of a system of linear equations. To prove the solvability of this system we use the Vandermonde determinant and its modification. Since this modification, as we know, does not appear in the literature we give the method of its calculus.

2. Formulate the corresponding problem of queueing theory. Let $\xi(t)$, $t \geq 0$ be a homogeneus Markov chain with state space $\{0,1,\ldots\}$ and transition probabilites for $\triangle$ $/\triangle \to 0/$:

$$\begin{aligned} P\{k \xrightarrow{\triangle} k-c+r\} &= \delta_{cr} + a_r \triangle + o(\triangle), \\ P\{k \xrightarrow{\triangle} r\} &= \delta_{ir} + b_{ir} \triangle + o(\triangle), \\ i &= 0,1,\ldots,c-1\ . \end{aligned} \tag{1}$$

Here $a_c < 0$, $b_{ii} < 0$; a_r $/r \neq 0/$, b_{ir} $/i \neq r/$ are nonnegative and the relations

$$(2) \qquad \sum_{r=0}^{\infty} a_r = 0, \ \sum_{r=0}^{\infty} b_{ir} = 0, \ i = 0, 1, \dots, c-1$$

hold. $\xi(t)$ may be interpreted as the number of costumers in a queueing system which is functioning by the following way:

1. if $\xi(t) = k \geq c$ then for $\triangle$ with probability $a_r \triangle + o(\triangle) / \ r = 0, 1, \dots, c-1$ / is completed the service of group consisting of $c-r$ requests, with probability $1 + a_c \triangle + o(\triangle)$ in the system there are not changes, and finally with probability $a_r \triangle + o(\triangle)$ / $r = c+1, c+2, \dots$ / a group consisting of $r-c$ requests enters the system;

2. if $\xi(t) = i$ $/1 \leq i \leq c-1/$ then for $\triangle$ with probability $b_{ir} \triangle + o(\triangle)$ $/r = 0, 1, \dots, i-1/$ is completed the service of a group consisting of $i-r$ reguests, with probability $1 + b_{ii} \triangle + o(\triangle)$ in the system there are no changes, and finally with probability $b_{ir} \triangle + o(\triangle)$ a group consisting of $r-i$ requests enters the system. In case $i = 0$ we have only the last two possibilities.

3. We derive the direct system of Kolomogorov differential equations for the transition probabilities $P_{lk}(t)$. According to (1)

$$P_{lk}(t+\triangle) = \sum_{i=0}^{c-1} P_{li}(t)[\delta_{ik} + b_{ik}\triangle + o(\triangle)] + \\ + \sum_{r=c}^{k+c} P_{lr}(t)[\delta_{rk} + a_{k-r+c}\triangle + o(\triangle)],$$

from which

$$(3) \qquad \frac{dP_{lk}(t)}{dt} = \sum_{r=0}^{c-1} P_{lr}(t) b_{rk} + \sum_{r=c}^{k+c} P_{lr}(t) a_{k-r+c}, \\ P_{lk}(0) = \delta_{lk}, \ k \geq 0 \ /l \geq 0 \text{ fixed}/.$$

Introduce the generating functions and Laplace transforms

$$
(4)\qquad
\begin{aligned}
P_l(t,\Theta) &= \sum_{k=0}^{\infty} P_{lk}(t)\Theta^k, \quad a(\Theta) = \frac{1}{\Theta^c}\sum_{k=0}^{\infty} a_k\Theta^k,\\
b_i(\Theta) &= \frac{1}{\Theta^i}\sum_{k=0}^{\infty} b_{ik}\Theta^k, \quad i = 0,1,\dots,c-1,\\
\tilde{P}_{lk}(s) &= \int_0^{\infty} e^{-st}P_{lk}(t)dt,\\
\tilde{P}_l(s,\Theta) &= \int_0^{\infty} e^{-st}P_l(t,\Theta)dt \ (0<|\Theta|\le 1, s>0).
\end{aligned}
$$

Using (3) and (4) we obtain

$$\frac{\partial P_l(t,\Theta)}{\partial t} = a(\Theta)P_l(t,\Theta) + \sum_{i=0}^{c-1} P_{li}(t)\Theta^i[b_i(\Theta)-a(\Theta)],$$

or

$$[s-a(\Theta)] = \tilde{P}_l(s,\Theta) = \Theta^l + \sum_{i=0}^{c-1}\tilde{P}_{li}(s)\Theta^i[b_i(\Theta)-a(\Theta)].$$

For each $s>0$ $\tilde{P}_l(s,\Theta)$ is bounded in the circle $|\Theta|<1$. So to find the unknown $\tilde{P}_{li}(s)$ it is natural to use the roots of the equation

$$(6)\qquad s - a(\Theta) = 0$$

in the circle $|\Theta|<1$, according to Rouché's theorem their number is equal to c. Denote them by $\lambda_1(s), \lambda_2(s), \dots, \lambda_c(s)$. If they all are simple substituting them into (5) we obtain a system of linear equations

$$\sum_{i=0}^{c-1} \tilde{P}_{li}(s)\lambda_j^i(s)[s - b_i(\lambda_j(s))] = \lambda_j^l(s),$$

$$j = 1, 2, \ldots, c$$

It is necessary to show that the determinant of (7) is different from zero. Since the investigated chain is regular, it is uniquely determined by its local characteristics. So there exist such $\tilde{P}_{l0}(s), \ldots, \tilde{P}_{l,c-1}(s)$ which satisfy (7), i.e. the system of equations is consistent for any initial state $l \geq 0$. Assume that one of equations (7) is the linear combination of others, i.e. there exist such constants $A_i / i = 1, \ldots, c/$ not all equal to zero, multiplying by which the elements of any column and summing them up we get zero. Because of the consistensy of the system the same assertion is valid for the free members for any $l \geq 0$.

Let l take on successivily the values $k, k+1, \ldots, k+c-1$. Then to determinate the constans A_i we obtain the following system of homogeneous linear equations

$$\begin{aligned} \lambda_1^k A_1 + \lambda_2^k A_2 + \ldots + \lambda_c^k A_c &= 0 \\ \lambda_1^{k+1} A_1 + \lambda_2^{k+1} A_2 + \ldots + \lambda_c^{k+1} A_c &= 0 \\ &\ldots \\ \lambda_1^{k+c-1} A_1 + \lambda_2^{k+c-1} A_2 + \ldots + \lambda_c^{k+c-1} A_c &= 0 \end{aligned} \tag{8}$$

whose determinant, after taking $\lambda_1^k \lambda_2^k \ldots \lambda_c^k$ out, will be the well-know Vandermonde determinante, so it is different from zero. So (8) has the unique solution $A_i = 0$, from which follows that the determinant of (7) is different from zero.

Now let (6) have n different roots with multiplicities r_1, $r_2, \ldots, r_n$ correspondingly, $\sum_{i=1}^{n} r_i = c$. In this case we use the fact if a is the root of equations $f'(x) = 0$ with multiplicity $r-1, \ldots$, and finally it is the simple root of equation $f^{(r-1)}(x) = 0$. So in the system of equations determining $\tilde{P}_{li}(s)$ to each different root will correspond $r_i / i = 1, 2, \ldots, n/$ equations, which we obtain from (5) by differentiation $0, 1, \ldots, r_i - 1$ times and substitution $\Theta = \lambda_i(s)$. In this case using the lemma concerning the modification of the Vandermonde determinant /see later/ one can make the same conclusion as in case of simple roots. We proved

Theorem 1.

$$\tilde{P}_l(s, \Theta) = \frac{\Theta^l + \sum_{i=0}^{c-1} \tilde{P}_{li}(s) \Theta^i [b_i(\Theta) - a(\Theta)]}{s - a(\Theta)}, \tag{9}$$

where $\tilde{P}_{l0}(s), \tilde{P}_{l1}(s), \ldots, \tilde{P}_{l,c-1}(s)$ in case of simple roots of (6) are determined by the system of equations (7), in case of roots with multiplicities by a system of equations obtained from (5) by differentation $0, 1, \ldots, r_j - 1$ times by Θ and substitution $\Theta = \lambda_j(s)$, where r_j is the multiplicity of th j-th root.

4.Determine the ergodic distribution for the described system. It is well-known that for a chain with communicating states the limits

$$\lim_{t \to \infty} P_{lk}(t) \geq 0$$

exist and do not depend on the initial state l. We find the conditions necessary and sufficient for the exist of the ergonic distribution.

According to [2] the necessary condition of existence of the ergodic distribution for bounded one-dimensional random walk is the difference from zero of the mean value of one step and its direction to the side of bound. For our system it means the fulfillment of the condition $a'(1) < 0$, what, according to Zyukov's results [2] implies the fact the root of (6) with maximal absolute value in the unit circle $\lambda_1(s)$, which obligatory simple, tends to $\lambda_1 = 1$ as $s \to 0$. Multiplying the both sides of (9) by s, as $s \to 0$

$$P(\Theta) = \sum_{k=0}^{\infty} P_k \Theta^k = \sum_{i=0}^{c-1} [1 - \frac{b_i(\Theta)}{a(\Theta)}] \Theta^i P_i, \ 0 < \Theta < 1.$$

Taking the limit as $\Theta \to 1$ we obtain

$$\int_{i=0}^{c-1} [1 - \frac{b_i'(1)}{a'(1)}]P_i = 1,$$

froom which follows $b_i'(1) < +\infty$.

We mention that to prove th sufficiency of the above conditions it is enough to consider the imbedded chain. Under these conditions the sufficiency follows from theorem [3]: Let $\{\xi_n, n \geq 0\}$ be homogeneus irreducible Markov chain. Assume that exits a nonnegative function $f(i)$, $i \geq 0$ with the following properties:

1. $M(f(\xi_{n+1})|\xi_n = i) < +\infty, \quad i \geq 0,$
2. $M(f(\xi_{n+1})|\xi_n = i) \leq f(i) - \epsilon, \quad i \geq N,$

where $\epsilon > 0$, $N \geq 0$ are fixed numbers. In this case all the states of the chain are ergodic. For our system let $N = c$ and $f(i) = i$. Then

$$1\frac{b_{01}}{-b_{00}} + 2\frac{b_{02}}{-b_{00}} + 3\frac{b_{03}}{-b_{00}} + \ldots = \frac{b_0'(1)}{-b_{00}} < \infty$$

$$0\frac{b_{10}}{-b_{11}} + 2\frac{b_{12}}{-b_{11}} + 3\frac{b_{13}}{-b_{11}} + \ldots = \frac{b_1'(1)}{-b_{11}} + 1 < \infty$$

$$\ldots$$

$$0\frac{b_{c-1,0}}{-b_{c-1,c-1}} + 1\frac{b_{c-1,1}}{-b_{c-1,c-1}} + \ldots = \frac{b_{c-1}'(1)}{-b_{c-1,c-1}} + c_1 < \infty$$

$$0\frac{a_0}{-a_c} + 1\frac{a_1}{-a_c} + 2\frac{a_2}{-a_c} + \ldots = \frac{a'(1)}{-a_c} + c < c - \epsilon$$

$$1\frac{a_0}{-a_c} + 2\frac{a_1}{-a_c} + \ldots = \frac{a'(1)}{-a_c} + c + 1 < c + 1 - \epsilon$$

which because of $a'(1) < 0$ and $b_i'(1) < \infty$ $/i = 0, 1, \ldots, c-1/$, implies the fullfillment of conditions of the theorem.

Let us denote by $D(s)$ the determinant of the system of equations to obtain the unknown $\tilde{P}_{li}(s)$ $/i = 0, 1, \ldots, c-1/$ /in case of simple roots it coincides with the determinant of (7). in case of multiple roots the elements of certain rows are replaced by the derivates of the corresponding elements of the previous according to the multiplicity of $\lambda_j(s)$ /. Furthermore, let $D_{il}(s)$ denote the determinant obtained from $D(s)$ replacing the $i+1$-st column $/i = 0, 1, \ldots, c-1$ / by the vector

$$\{\lambda_1^l(s), \lambda_2^l, \ldots, \lambda_c^l(s)\}^T \tag{10}$$

in case of simple roots /T means the operation of transposition/, and by replacement with the vector got from (10) by means of the corresponding deifferentiation in case of multiple roots. In this case it is valid the following

Theorem 2. *The chain $\xi(t)$, $t \geq 0$ is ergodic if and only if 1. all the states are communicating; 2. $a'(1) < 0$; 3. $b_i'(1) < +\infty/i = 0, 1, \ldots, c-1/$. The generating function of ergodic distribution has the form*

$$P(\Theta) = \sum_{i=0}^{c-1} [1 - \frac{b_i(\Theta)}{a(\Theta)}] \Theta^i P_i,$$

where

$$P_i = \lim_{s \to 0} \frac{s D_{il}(s)}{D(s)}.$$

5.We are going to consider a modification of the well-known Vandermonde determinant / only the elements of the first row have the power m /. Let be given a sequence of integers $r_1, \ldots, r_n$, $\sum_{k=1}^{n} r_k = c$. The first column of the determinant we take from the original Vandermonde determinant, the second column is replaced by the first derivates of the elements of the first column, the third column by the second derivates, ..., and, finally the r_1-th column by the $r_1 - 1$-th derivates. The $r_1 + 1$-st column is taken again from the Vandermonde determinant and an analogous replacement is made /only one uses r_2 instead of r_2 /. This procedure is realized n times. We prove the

Lemma.

(11)

$$\left|\begin{matrix} x_1^m & mx_1^{m-1} & m(m-1)x_1^{m-2} & \dots & x_n^m & \dots \\ x_1^{m+1} & (m+1)x_1^m & (m+1)mx_1^{m-1} & \dots & x_n^{m+1} & \dots \\ x_1^{m+2} & (m+2)x_1^{m+1} & (m+2)(m-1)x_1^m & \dots & x_n^{m+2} & \dots \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ x_1^{m+c-1} & (m+x-1)x_1^{m+c-2} & \dots & \dots & x_n^{m+c-1} & \dots \end{matrix}\right|$$

$$= \prod_{k=1}^{n} \prod_{l=0}^{r_k-1} (l!) \prod_{k=1}^{n} x_x^{mr_k} \prod_{1\le i\le j\le n} (x_j - x_i)^{r_i r_j},$$

where n is the number of different elements x_k, $1 \le k \le n$,r_k is the number of columns containing x_k, $\sum_{k=1}^{n} r_k = c$.

Proof. (11) will be considered as the function of arguments $x_1, \dots, x_n$. Consider columns containing a fixed variable x_k and take outside the sign of the determinant $0!x_k^m$ from the first column, $1!x_k^{m-1}$ from the second one,...,and finally $(r_k-1)!x_k^{m-r_k+1}$ from the r_k-th one. Using the same procedure for all n variables we obtain the factor

$$\prod_{k=1}^{n} \prod_{l=0}^{r_k-1} (l!) \prod_{k=1}^{n} x_k^{mr_k - \frac{r_k(r_k-1)}{2}} \tag{12}$$

and the determinant to be computed will consist of stripes of the form

(13)

$$\begin{bmatrix} 1 & C_m^{m-1} & C_m^{m-2} & \dots & C_m^{m-r_k+1} \\ x_k & C_{m+1}^{m} x_k & C_{m+1}^{m-1} x_k & \dots & C_{m+1}^{m-r_k+2} x_k \\ x_k^2 & C_{m+2}^{m+1} x_k^2 & C_{m+2}^{m} x_k^2 & \dots & C_{m+2}^{m-r_k+3} x_k^2 \\ \vdots & \vdots & \vdots & & \vdots \\ x_k^{c-1} & C_{m+c-1}^{m+c-2} x_k^{c-1} & C_{m+c-1}^{m+c-3} x_k^{c-1} & \dots & C_{m+c-1}^{m+c-r_k} x_k^{c-1} \end{bmatrix}.$$

Now subtract the first column from the second, the new second column from the third one,...,and at last the $r_k - 1$-st from the r_k-th. Executing this procedure m times using the equalities $C_n^n = C_{n-1}^{n-1}$ and $C_n^k = C_{n-1}^k + C_{n-1}^{k-1}$, taking out from the second column $x_k, \dots,$ from th r_k-th one $x_k^{r_k - 1}$ (13) takes on the form

$$(14) \qquad \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ x_k & C_1^0 & 0 & \dots & 0 \\ x_k^2 & C_2^1 x_k & C_2^0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ x_k^{c-1} & C_{c-1}^{c-2} x_1^{c-2} & C_{c-1}^{c-3} x_1^{c-3} & \dots & C_{c-1}^{c-r_k} x_k^{c-r_k} \end{bmatrix}.$$

(12) will be multiplied by $\prod_{k=1}^n x_k^{\frac{r_k(r_k-1)}{2}}$ and takes on the form

$$\prod_{k=1}^{n} \prod_{l=0}^{r_k - 1} (l!) \prod_{k=1}^{n} x_k^{m r_k}.$$

We show that the value of the detereminant composed from stripes of form (14) depends only on the difference of the arguments. Let α be an arbitrary number and x any of $x_1, \dots, x_n$.Consider the $n+1$-st row. In the first column one finds in this place x^n. We add to this row the first one multipled by $C_n^0 \alpha^n$, the second one multiplied by $C_n^1 \alpha^{n-1}, \dots,$ and finally the n-th row multiplied by $C_n^{n-1} \alpha$. Obviously, on the place of x^n we get the value $(x+\alpha)^n$, and the value of the determinant remains unchanged.

Now we consider the case of columns got by means of differentiation. Fix the row with number $n+1$ and consider the row with number $n+1-i$ and the column got from the first one after differentiation j times.So on the intersection of the $n+1$-st row and the $j+1$-st column stands $C_j^{n-j} x^{n-j}$. We make the same procedure as in case of the first column. i may change from 0 till $n-j$, in the column above the element $C_n^{n-j} x^{n-j}$ stand the elements $C_{n-i}^{n-j-i} x^{n-j-i}$. Since the $n+1-i$-th row is multiplied by $C_n^{n-i}\alpha^i$, so using the equality

$$C_n^{n-i}C_{n-i}^{n-j-i} = C_n^{n-j}C_{n-j}^i$$

on the place of $C_n^{n-j}x^{n-j}$ we get the value

$$\sum_{i=0}^{n-j} C_{n-i}^{n-j-i}x^{n-j-i}C_n^{n-i}\alpha^i = C_n^{n-j}\sum_{i=0}^{n-j} C_{n-j}^i x^{n-j-i}\alpha^i = \\ = C_n^{n-j}(x+\alpha)^{n-j},$$

which proves our assertion.

Suppose the columns of our original determinant are arrenged according to the multiplicities of the variables, i.e. the condition $r_1 \geq r_2 \geq \ldots \geq r_n$ is fullfilled. It can be easily seen that there is no loss of generality since after a certain number of transposition the determinant always may be represented in such form. The fact that the sign of the determinant does not change follows from the proof and from the condition that in the differences $x_j - x_i$ always $j > i$. We prove our assertion by induction. Let $n = 2$. Then on the basis of (14) our determinant takes on the form

$$(15)\qquad \begin{vmatrix} 1 & 0 & \ldots & 1 & \ldots \\ x & C_1^0 & \ldots & y & \ldots \\ x^2 & C_2^1 x & \ldots & y^2 & \ldots \\ x^3 & C_3^2 x^2 & \ldots & y^3 & \ldots \\ x^{r_1+r_2-1} & C_{r_1+r_2-1}^{r_1+r_2+2} x^{r_1+r_2-2} & \ldots & y^{r_1+r_2-1} & \ldots \end{vmatrix}.$$

We have shown that the value of this determinant remains unchanged if we add arbitrary number to its variables, so we subtract from all the variables x. In this case in the first r_1 columns only the elements of the principal diagonal are different from zero, they all are equal to unity. So we have to evaluate the value of the determinant

$$\begin{vmatrix} C_{r_1}^{r_1}(y-x)^{r_1} & \ldots & C_{r_1}^{r_1-r_2+1}(y-x)^{r_1-r_2+1} \\ C_{r_1+1}^{r_1+1}(y-x)^{r_1+1} & \ldots & C_{r_1+1}^{r_1-r_2+2}(y-x)^{r_1-r_2+2} \\ C_{r_1+r_2-1}^{r_1+r_2-1}(y-x)^{r_1+r_2-1} & \ldots & C_{r_1+r_2-1}^{r_1}(y-x)^{r_1} \end{vmatrix}.$$

Take out from the first column $(y-x)^{r_1}$,$(y-x)^{r_1-1}$ from the second, ... $(y-x)^{r_1-r_2+1}$ from the last, i.e. from the whole determinant

$$(y-x)^{r_1 r_2 - \frac{r_2(r_2-1)}{2}}.$$

After this operation in each row the elements belonging to the different columns contain $y-x$ at the same power. Subtract from the second column the first, from the third the second, etc. Executing this procedure r_1 times we obtain

$$\begin{vmatrix} C_0^0 & 0 & \dots & 0 \\ C_1^1(y-x) & C_1^0(y-x) & \dots & 0 \\ \vdots & \vdots & & \vdots \\ C_{r_2-1}^{r_2-1}(y-x)^{r_2-1} & C_{r_2-1}^{r_2-1}(y-x)^{r_2-1} & \dots & C_{r_2-1}^{0}(y-x)^{r_2-1} \end{vmatrix}.$$

Factor out from the second column $y-x$, from the third one $(y-x)^2$,... ,and, finally, from the last column $(y-x)^{r_2-1}$ i.e. from the whole determinant

$$(y-x)^{\frac{r_2(r_2-1)}{2}}.$$

After such transformations we obtain a determinant in which all elements above the principal diagonal are equal to zero and all elements on the principal diagonal are equal to unity. Collecting all our results we get that (15) is equal to $(y-x)^{r_1 r_2}$ what proves the lemma in the case of two variables.

Assume the lemma is proved in case of $n-1$ variables and prove it for the case of n ones.Since the value of the determinant depends only on the differences of the arguments instead of $x_1,\dots,x_n$ we put $x_1 - x_1 = 0$, $x_2 - x_1,\dots,x_n - x_1$. Now in columns which contained x_1 the elements of the principal diagonal are equal to unity, all other elements are equal to zero. So we have to find the value of determinant

$$\begin{vmatrix} (x_2-x_1)^{r_1} & C_{r_1}^{r_1-1}(x_2-x_1)^{r_1-1} & \dots \\ (x_2-x_1)^{r_1+1} & C_{r_1+1}^{r_1}(x_2-x_1)^{r_1} & \dots \\ \vdots & \vdots & \vdots \\ (x_2-x_1)^{r_1+\dots+r_n+1} & C_{r_1+\dots+r_n-1}^{r_1+\dots+r_n-2}(x_2-x_1)^{r_1+\dots+r_n-2} & \dots \end{vmatrix}.$$

From this determinant, similarly to the case of two variables, the multipliers

$$(x_j - x_1)^{r_1 r_j - \frac{r_j(r_j-1)}{2}}, \quad j = 2, 3, \ldots, n,$$

can be factored out. Executing the same procedure concerning the subtraction of columns and factoring out the corresponding multipliers, from the whole determinant we get the factors

$$(x_2 - x_1)^{r_1 r_2}, (x_3 - x_1)^{r_1 r_3}, \ldots, (x_n - x_1)^{r_1 r_n}.$$

If we denote the determinant corresponding to (14) by $D_n(x_1, \ldots, x_n)$ then obtain the following result

$$D_n(x_1, \ldots, x_n) = D_{n-1}(x_2 - x_1, \ldots, x_n - x_1) \prod_{j=2}^{n} (x_j - x_1)^{r_1 r_j} =$$

$$= D_{n-1}(x_2, \ldots, x_n) \prod_{j=2}^{n} (x_j - x_1)^{r_1 r_j},$$

where for $D_{n-1}(x_2, \ldots, x_n)$ an analogus representation is valid according to the assumption of induction. The lemma is proved.

REMARK. We can consider a similar determinant whose evalution leads to the previous one. We show that

$$\begin{vmatrix} \frac{1}{x_1^m} & -\frac{m}{x_1^{m+1}} & \frac{(m+1)m}{x_1^{m+2}} & \cdots & \frac{1}{x_n^m} & \cdots \\ \frac{1}{x_1^{m+1}} & -\frac{m+1}{x_1^{m+2}} & \frac{(m+2)(m+1)}{x_1^{m+3}} & \cdots & \frac{1}{x_n^{m+1}} & \cdots \\ \frac{1}{x_1^{m+2}} & -\frac{m+2}{x_1^{m+3}} & \frac{(m+3)(m+2)}{x_1^{m+4}} & \cdots & \frac{1}{x_n^{m+2}} & \cdots \\ \vdots & \vdots & \vdots & & \vdots & \\ \frac{1}{x_1^{m+c-1}} & -\frac{m+c-1}{x_1^{m+c}} & \frac{(m+c)(m+c-1)}{x_1^{m+c+1}} & \cdots & \frac{1}{x_n^{m+c-1}} & \cdots \end{vmatrix} =$$

$$= (-1)^{\sum_{k=1}^{n} [\frac{r_k}{2}]} \prod_{k=1}^{n} \prod_{l=0}^{r_k-1} (l!) \prod_{k=1}^{n} x_k^{-r_k(m+r_k-1)}.$$

$$\cdot \prod_{1 \le i < j \le n} \left(\frac{1}{x_j} - \frac{1}{x_i}\right)^{r_i r_j},$$

where n is the number of different variables, r_k the number of columns containing x_k, and $[\frac{r_k}{2}]$, as usually denotes the greatest integer in $\frac{r_k}{2}$, $\sum_{k=1}^{n} r_k = c$.

Proof. First of all we remark that for each variable $x_i/i = 1,2,\ldots,n/$ the elements of each even column have the sign minus /it follows from the fact that we differentiate variables with negative exponents/. So for each variable -1 is factored out in the power $[\frac{r_k}{2}]$. Futhermore, for each variable we factor out from the first column $0!\frac{1}{x_i^m}$, from the second one $1!\frac{1}{x_i^{m+1}},\ldots$, from the last one$(r_i - 1)!\frac{1}{x_i^{m+r_i-1}}$. After such operations we obtain the factor

$$(-1)^{\sum_{i=1}^{n}[\frac{r_i}{2}]} \prod_{i=1}^{n} \prod_{l=0}^{r_i-1} (l!) \prod_{i=1}^{n} x_i^{-\frac{r_i(2m+r_i-1)}{2}},$$

and the determinant takes on the form

$$\begin{vmatrix} 1 & C_m^{m-1} & C_{m+1}^{m-1} & \ldots & 1 & \ldots \\ \frac{1}{x_1} & C_{m+1}^{m}\frac{1}{x_1} & C_{m+2}^{m}\frac{1}{x_1} & \ldots & \frac{1}{x_n} & \ldots \\ \frac{1}{x_1^2} & C_{m+2}^{m+1}\frac{1}{x_1^2} & C_{m+3}^{m+1}\frac{1}{x_1^2} & \ldots & \frac{1}{x_n^2} & \ldots \\ \frac{1}{x_1^{c-1}} & C_{m+c-1}^{m+c-2}\frac{1}{x_1^{c-1}} & C_{m+c}^{m+c-2}\frac{1}{x_1^{c-1}} & \ldots & \frac{1}{x_n^{c-1}} & \ldots \end{vmatrix}.$$

Using the equality $C_n^k = C_{n-1}^k + C_{n_1}^{k+1}$ the elements of the last column for each variable may be represented in the form of a sum. Subtract from the last column the elements of the previous one. We execute this procedure $r_j - 2$ times /i.e. subtract from the $i+1$-st column the i-th one, the so obtained $i+1$-st column from the $i+2$-nd, etc., and the transformation in each step begins with the column standing one place more left /. Then after subtitution

$\frac{1}{x_j} = y_j$ we come to

$$\begin{vmatrix} \dots & 1 & C_m^{m-1} & \dots & C_m^{m-r_j+1} & \dots \\ \dots & y_j & C_{m+1}^{m} y_j & \dots & C_{m+1}^{m-r_j+2} y_j & \dots \\ \dots & y_j^2 & C_{m+2}^{m+1} y_j^2 & \dots & C_{m+2}^{m-r_j+3} y_j^2 & \dots \\ & \vdots & \vdots & & \vdots & \\ \dots & y_j^{c-1} & C_{m+c-1}^{m+c-2} y_j^{c-1} & \dots & C_{m+c-1}^{m+c-r_j} y_j^{c-1} & \dots \end{vmatrix}$$

i.e. to a determinant of form (13). After analogous transformations

$$y_j^{\frac{r_j(r_j-1)}{2}} = x_j^{-\frac{r_j(r_j-1)}{2}}$$

can be taken out. Collecting all factors we obtain the final result

$$(-1)^{\sum_{i=1}^{n}[\frac{r_i}{2}]} \prod_{i=1}^{n} \prod_{l=0}^{r_i-1} (l!) \prod_{i=1}^{n} x_i^{-r_i(m+r_i-1)} \prod_{1 \le i < j \le n} (\frac{1}{x_j} - \frac{1}{x_i})^{r_i r_j}.$$

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