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ON THE DISCRETE LYAPUNOV PROBLEM

S.E. GYARMATI

It is well-known that constrained extremal problems of Lyapunov type play an important role in optimization theory. In particular, some optimal control problems can be reduced to a Lyapunov problem, see e.g. [1].

In this work we are concerned with an infinite time discrete version of the Lyapunov problem. Since the usual methods of the continuous case cannot be applied to our problem, suitable generalized measure-valued controls are introduced. The main result gives a necessary and sufficient condition for the optimality of a process. An existence theorem is also obtained.

1. Let $k \in \mathbf{N}$ be a natural number and $U \subset \mathbf{R}^k$ a non-empty compact set. Denote by $C(U)$ the set of all real-valued continuous functions defined on U. $C(U)$ is a separable Banach space with respect to the usual operations and the norm $\| f \| := max\{| f(x) | \mid x \in U\}$ $(f \in C(U))$. We introduce the following notations:

$$l_1\,[C(U)] := \{h : \mathbf{N} \to C(U) \mid \sum_{t=1}^{\infty} \| h(t) \| < \infty\},$$

$$l_\infty\,[C(U)^*] := \{\mu : \mathbf{N} \to C(U)^* \mid sup\{\| \mu(t) \|_* \mid t \in \mathbf{N}\} < \infty\},$$

where $C(U)^*$ is the topological dual of $C(U)$ with the usual norm $(\| \|_*)$. If we take in $l_1[C(U)]$ and $l_\infty[C(U)^*]$ the norms

$$\| h \|_1 := \sum_{t=1}^{\infty} \| h(t) \| \quad (h \in l_1[C(U)]),$$

resp.

$$\| \mu \|_\infty := \sup\{\| \mu(t) \|_* \in \mathbf{R}_\circ^+ \mid t \in \mathbf{N}\} \quad (\mu \in l_\infty[C(U)^*])$$

then $l_1[C(U)]$ is a separable Banach-space, while $l_\infty[C(U)*]$ is a Banach space.

Let us introduce the following mapping

$$F : l_\infty[C(U)^*] \to l_1[C(U)]^*,$$

$$\langle F(\mu), h \rangle := \sum_{t=1}^{\infty} \langle \mu(t), h(t) \rangle \quad (\mu \in l_\infty[C(U)^*], h \in l_1[C(U)]).$$

It is easy to prove that F is an isometrical isomorphism between the just mentioned spaces. Thus, in what follows, we shall identify $l_\infty[C(U)^*]$ and $l_1[C(U)]^*$.

Denote by $\tilde{U}$ the set of all probability measures defined on U. The elements of the set

$$\mathcal{U} := \{\mu \in l_\infty[C(U)^*] \mid \mathcal{R}_\mu \subset \tilde{U}\}$$

are called *generalized controls.* We shall assume that the set

$$\mathcal{U} \subset l_\infty[C(U)^*] = l_1[C(U)]^*$$

is endowed with the weak-star topology of $l_1[C(U)]^*$.

The first statement contains the basic properties of U.

Theorem 1. *The set of the generalized controls is a convex compact subset of* $l_\infty[C(U)^*]$.

Proof. By the definition of a probability measure, the convexity of U follows immediately. On the other hand, $l_1[C(U)]$ is separable. Thus, by means of the known theorems of Bishop and Alaoglu /see [2] I.3.11., I.3.12./ we get that the set

$$B\left(l_\infty\left[C(U)^*\right]\right) := \{\mu \in l_\infty\left[C(U)^*\right] \mid \|\mu\|_\infty \leq 1\}$$

is compact and metrizable with respect to the weak-star topology. Since $\mathcal{U} \subset B\left(l_\infty\left[C(U)^*\right]\right)$, it is enough to prove that U is sequentially compact in the mentioned topology.

To this end let $(\mu_n) : \mathbf{N} \to \mathcal{U}$ be a weak-star convergent sequence and $\lim(\mu_n) =: \nu \in B\left(l_\infty\left[C(U)^*\right]\right)$. Taking into consideration the mapping F, it follows that, for all $n \in \mathbf{N}$, μ_n can be identified with an element of $l_1[C(U)]^*$. Hence, if $h \in l_1[C(U)]$ then $\lim(\langle\mu_n, h\rangle) = \langle\nu, h\rangle$. For all $t \in \mathbf{N}$ and $\varphi \in C(U)$ let $h \in l_1[C(U)]$ be defined as follows

$$h(n) := \begin{cases} \varphi, & \text{if } \ n = t \\ 0, & \text{if } \ n \in \mathbf{N} \setminus \{t\} \end{cases} \quad (n \in \mathbf{N}).$$

Then $\langle\mu_n, h\rangle = \langle\mu_n(t), \varphi\rangle$ and $\langle\nu, h\rangle = \langle\nu(t), \varphi\rangle$, from which, by the weak-star convergence, we obtain that

$$\lim\left(\langle\mu_n(t), \varphi\rangle\right)_{n\in\mathbf{N}} = \langle\nu(t), \varphi\rangle.$$

If $\varphi \in C(U)$ and $\varphi(u) \geq 0$ then $\int_U \varphi d\mu_n(t) \geq 0$ for all $t \in \mathbf{N}$. Since $\lim(\mu_n) = \nu$ therefore $\int_U \varphi d\nu(t) \geq 0$, i.e. $\nu(t)$ is a non-negative measure for all $t \in \mathbf{N}$ /[2] I.5.5. (1)/.

Now we take the special case $\varphi(u) = 1 (u \in U)$ which leads to

$$\nu(t)(U) = \int_U 1 d\nu(t) = \langle\nu(t), \varphi\rangle = \lim(\langle\mu_n(t), \varphi\rangle)_{n\in\mathbf{N}} =$$
$$= \lim(\mu_n(t)(U))_{n\in\mathbf{N}} = \lim(1) = 1.$$

Hence $\nu \in U$ as it was stated.

2. Let $m \in \mathbf{N}$ and $\varphi_i \in l_1[C(U)](i \in \overline{0,m})$ be fixed. Define the functions $\mathcal{F}_i : \mathcal{U} \to \mathbf{R}(i \in \overline{0,m})$ in the following way :

$$\mathcal{F}_i(\mu) := \sum_{\tau=1}^{\infty} \int_{\mathcal{U}} \varphi_i(\tau) d\mu(\tau) \quad (i \in \overline{0,m}).$$

We notice that this definition is correct since the series in the above equality is absolute convergent. Furthermore, let X be an arbitrary linear space over the real field $\mathbf{R}$, $X_0 \subset X$ a non-empty convex subset and $m', m \in \mathbf{N}, m' \leq m$. Let $g_i : X_0 \to \mathbf{R}(i \in \overline{0,m})$ be given functions which are convex for $i \in \overline{0,m'}$ and affine for $i \in \overline{m'+1,m}$. Using these notations we can formulate the so-called *discrete Lyapunov-problem* in the following way:

minimize the function

$$(x,\mu) \mapsto \mathcal{F}_0(\mu) + g_0(x) \quad ((x,\mu) \in X_0 \times \mathcal{U})$$

on the set

$$\mathcal{D} := \{(x,\mu) \in X_0 \times \mathcal{U} \mid \mathcal{F}_i(\mu) + g_i(x) \leq 0 \quad (i \in \overline{1,m'}),$$
$$\mathcal{F}_i(\mu) + g_i(x) = 0 \quad (i \in \overline{m'+1,m})\}.$$

The elements of D are called *admissible processes.* A solution $(\widehat{x}, \widehat{\mu})$ to the above problem is called an *optimal process.* In the following theorem we give a sufficient condition for the existence of a solution of the problem.

Theorem 2. *Suppose that the above X is a topological vector space, X_0 is compact, $\mathcal{D} \neq \emptyset, g_i$ is lower semi-continuous for $i \in \overline{0,m'}$ and continuous for $i \in \overline{m'+1,m}$. Then there exists a solution of the discrete Lyapunov problem.*

Proof. From the definition of $\mathcal{F}_i$ and the assumptions on $g_i(i \in \overline{0,m})$ we get that the mappings

$$X_0 \times \mathcal{U} \ni (x,\mu) \mapsto \mathcal{F}_i(\mu) \quad (i \in \overline{0,m}),$$
$$X_0 \times \mathcal{U} \ni (x,\mu) \mapsto g_i(x) \quad (i \in \overline{m'+1,m}),$$
$$X_0 \times \mathcal{U} \ni (x,\mu) \mapsto \mathcal{F}_i(\mu) + g_i(x) \quad (i \in \overline{m'+1,m})$$

are continuous. Similarly, it follows that the functions

$$X_0 \times \mathcal{U} \ni (x,\mu) \mapsto g_i(x) \quad (i \in \overline{0,m'})$$

are lower semi-continuous. This implies that the sets

$$\mathcal{D}_1 := \{(x,\mu) \in X_0 \times \mathcal{U} \mid \mathcal{F}_i(\mu) + g_i(x) \le 0,\ i \in \overline{1,m'}\}$$
$$\mathcal{D}_2 := \{(x,\mu) \in X_0 \times \mathcal{U} \mid \mathcal{F}_i(\mu) + g_i(x) = 0,\ i \in \overline{m'+1,m}\}$$

are closed, from which the closedness of $\mathcal{D} = \mathcal{D}_1 \cap \mathcal{D}_2$ follows. Furthermore, the set $X_0 \times \mathcal{U}$ is compact with respect to the product topology /see the assumptions of the theorem and Theorem 1/. Therefore, the closed set $\mathcal{D} \subset X_0 \times \mathcal{U}$ is compact too. Applying the classical Weierstrass theorem we get that the function $(x,\mu) \mapsto \mathcal{F}_0(\mu) + g_0(x)$ $((x,\mu) \in \mathcal{D})$ attains a minimum.

This proves Theorem 2.

3. In this section, following the usual method, we shall give a necessary and sufficient condition for the optimality of a process.

Theorem 3.

i/ *If* $(\widehat{x},\widehat{\mu}) \in X_0 \times \mathcal{U}$ *is a solution of the discrete Lyapunov-problem, then there exist* $\widehat{\lambda}_0 \in \mathbf{R}$ *and* $\widehat{\lambda} = (\widehat{\lambda}_1,\dots,\widehat{\lambda}_m) \in \mathbf{R}^m$ *such that* $\prod_{i=0}^{m} \widehat{\lambda}_i \neq 0$ *and*

$$\min\{\sum_{i=0}^{m} \widehat{\lambda}_i \int_U \varphi_i(\tau)d\mu(\tau) \mid \mu \in \mathcal{U}\} = \tag{1}$$
$$= \sum_{i=0}^{m} \widehat{\lambda}_i \int_U \varphi_i(\tau)d\widehat{\mu}(\tau) \quad (\tau \in \mathbf{N}),$$

$$\min\{\sum_{i=0}^{m} \widehat{\lambda}_i g_i(x) \mid x \in X_0\} = \sum_{i=0}^{m} \widehat{\lambda}_i g_i(\widehat{x}), \tag{2}$$

(3) $$\widehat{\lambda}_i \geq 0 \quad (i \in \overline{0,m'}),$$

(4) $$\widehat{\lambda}_i[\mathcal{F}_i(\widehat{\mu}) + g_i(\widehat{x})] = 0 \quad (i \in \overline{1,m}).$$

ii/ *If* $(\widehat{x},\widehat{\mu}) \in X_0 \times \mathcal{U}$ *and there are* $\widehat{\lambda}_0 > 0$ *and* $\widehat{\lambda} \in \mathbf{R}^m$ *such that* (1)-(4) *are true, then* $(\widehat{x},\widehat{\mu})$ *is optimal* .

Proof.

i/ First we prove the necessity. To this end we introduce the following functions:

$$\tilde{\mathcal{F}} : l_\infty[C(U)^*] \to \mathbf{R}^{m+1}$$

$$\tilde{\mathcal{F}}(\mu) := \left(\sum_{\tau=1}^{\infty}\int_U \varphi_i(\tau)d\mu(\tau)\right)_{i=0}^{m} :=$$

$$:= \left(\sum_{\tau=1}^{\infty}\int_U \varphi_0(\tau)d\mu(\tau),\ldots,\sum_{\tau=1}^{\infty}\int_U \varphi_m(\tau)d\mu(\tau)\right),$$

$$\mathcal{F} := \tilde{\mathcal{F}}\,|_{\mathcal{U}} = (\mathcal{F}_0,\ldots,\mathcal{F}_m).$$

It is clear that $\tilde{\mathcal{F}}$ is linear, therefore the image $\mathcal{R}_{\mathcal{F}}$ of the convex set U is convex.

Since $(\widehat{x},\widehat{\mu}) \in X_0 \times \mathcal{U}$ is a solution, for all $(x,\mu) \in \mathcal{D}$, we have

$$\mathcal{F}_0(\widehat{\mu}) + g_0(\widehat{x}) \leq \mathcal{F}_0(\mu) + g_0(x).$$

Without loss of generality we can assume that

(5) $$\mathcal{F}_0(\widehat{\mu}) + g_0(\widehat{x}) = 0.$$

For the proof we need the following set

$$C := \{\alpha = (\alpha_0,\ldots,\alpha_m) \in \mathbf{R}^{m+1} \mid \exists (x,\mu) \in \mathcal{D},$$
$$\mathcal{F}_0(\mu) + g_0(x) < \alpha_0; \mathcal{F}_i(\mu) + g_i(x) \leq \alpha_i \quad (i \in \overline{1,m'});$$
$$\mathcal{F}_i(\mu) + g_i(x) = \alpha_i \quad (i \in \overline{m'+1,m})\}.$$

First of all we prove that C is convex. Indeed, if we take $\alpha^1 = (\alpha_0^1, \ldots, \alpha_m^1), \alpha^2 = (\alpha_0^2, \ldots, \alpha_m^2) \in C$ and $\Theta \in]0,1[$ then there are $(x^1, \mu^1), (x^2, \mu^2) \in \mathcal{D}$ such that for $k \in \overline{1,2}$ we have

$$\mathcal{F}_0(\mu^k) + g_0(x^k) < \alpha_0^k,$$

$$\mathcal{F}_i(\mu^k) + g_i(x^k) \begin{cases} \geq \alpha_i^k, & \text{if} \quad i \in \overline{1,m'}\ , \\ = \alpha_i^k, & \text{if} \quad i \in \overline{m'+1,m}\ . \end{cases}$$

On the other hand, the convexity of $\mathcal{R}_{\mathcal{F}}$ provides an element $\mu_\Theta \in \mathcal{U}$ for which

$$\mathcal{F}_i(\mu_\Theta) = \Theta \mathcal{F}_i(\mu^1) + (1-\Theta)\mathcal{F}_i(\mu^2) \quad (i \in \overline{0,m}). \tag{6}$$

Furthermore, since X_0 is a convex set thus there is an $x_\Theta \in X_0$ with $x_\Theta = \Theta x^1 + (1-\Theta)x^2$. Taking into account our assumptions on the functions $g_i (i \in \overline{0,1})$, if follows that

$$g_i(x_\Theta) \leq \Theta g_i(x^1) + (1-\Theta)g_i(x^2) \quad (i \in \overline{0,m'}), \tag{7}$$

$$g_i(x_\Theta) = \Theta g_i(x^1) + (1-\Theta)g_i(x^2) \quad (i \in \overline{m'+1,m}). \tag{8}$$

On the basis of (6), (7) and (8) we get

$$\mathcal{F}_0(\mu_\Theta) + g_0(x_\Theta) < \Theta\alpha_0^1 + (1-\Theta)\alpha_0^2,$$

$$\mathcal{F}_i(\mu_\Theta) + g_i(\Theta) \begin{cases} \leq \Theta\alpha_i^1 + (1-\Theta)\alpha_i^2 & (i \in \overline{1,m'}) \\ = \Theta\alpha_i^1 + (1-\Theta)\alpha_i^2 & (i \in \overline{m'+1,m}), \end{cases}$$

i.e. $\Theta\alpha^1 + (1-\Theta)\alpha^2 \in C$. This proves the convexity of C.

Now we take into consideration that $(\widehat{x}, \widehat{\mu})$ is a solution, (5) implies that $(\alpha_0, \ldots, \alpha_m) \in C$ for all $\alpha_0 > 0, \alpha_i < 0$, $(i \in \overline{1,m'})$, $\alpha_i = 0 (i \in \overline{m'+1,m})$. In particular $C \neq \emptyset$.

We know that $0 \notin C$, therefore 0 can be separated from C. This means that there exist $\widehat{\lambda}_i \in \mathbf{R}(i \in \overline{0,m})$ such that $\prod_{i=0}^{m} \widehat{\lambda}_i \neq 0$ and

$$\sum_{i=0}^{m} \widehat{\lambda}_i \alpha_i \geq 0 \ (\alpha \in C). \tag{9}$$

Let ε be an arbitrary positive number and $j \in \overline{1,m'}$. Clearly

$$\alpha := (\underbrace{\varepsilon}_{0}, \underbrace{0}_{1}, \ldots, \underbrace{1}_{j}, \ldots 0) \in C.$$

Thus, by (9) we have

$$\sum_{i=1}^{m} \widehat{\lambda}_i \alpha_i = \varepsilon \widehat{\lambda}_0 + \widehat{\lambda}_j \geq 0.$$

Now we consider the following non-negative function defined on the set of the positive numbers

$$0 < \varepsilon \mapsto \varepsilon \widehat{\lambda}_0 + \widehat{\lambda}_j.$$

For this function we have

$$\lim_{0} (\varepsilon \mapsto \varepsilon \widehat{\lambda}_0 + \widehat{\lambda}_j) = \widehat{\lambda}_j \geq 0 \ \ (j \in \overline{1,m'}). \tag{10}$$

If ε stands again for a positive number, then $\alpha := (\varepsilon, 0, \ldots, 0) \in C$. Thus we get $\sum_{i=0}^{m} \widehat{\lambda}_i \alpha_i = \varepsilon \widehat{\lambda}_0 \geq 0$, i.e. $\widehat{\lambda}_0 \geq 0$. Hence, by (10), $\lambda_i \geq 0$ $(i \in \overline{0,m'})$, as it was stated in (3).

Now we recall again that $(\widehat{x}, \widehat{\mu}) \in \mathcal{D}$ is an admissible , i.e.

$$\mathcal{F}_i(\widehat{\mu}) + g_i(\widehat{x}) \begin{cases} \leq 0 & (i \in \overline{1,m'}), \\ = 0 & (i \in \overline{m'+1,m}). \end{cases} \tag{11}$$

This means that for all $\varepsilon > 0$

$$\alpha := (\underbrace{\varepsilon}_{0}, \underbrace{0}_{1}, \dots, \underbrace{\mathcal{F}_i(\widehat{\mu}) + g_i(\widehat{x})}_{i}, 0, \dots, \underbrace{0}_{m}) \in C \quad (i \in \overline{1,m}).$$

By the separation theorem applied above we get

$$\sum_{j=0}^{m} \widehat{\lambda}_j \alpha_j = \varepsilon \widehat{\lambda}_0 + \widehat{\lambda}_i [\mathcal{F}_i(\widehat{\mu}) + g_i(\widehat{x})] \geq 0 \quad (i \in \overline{1,m}).$$

As above, we consider the function

$$0 < \varepsilon \mapsto \widehat{\lambda}_0 \varepsilon + \widehat{\lambda}_i [\mathcal{F}_i(\widehat{\mu}) + g_i(\widehat{x})] \quad (i \in \overline{1,m})$$

and compute its limit at zero, which leads to

$$\widehat{\lambda}_i [\mathcal{F}_i(\widehat{\mu}) + g_i(\widehat{x})] \geq 0 \quad (i \in \overline{1,m}).$$

Hence, to prove (4) it is enough to refer to (11) and (3). Thus we obtain that

$$\widehat{\lambda}_i [\mathcal{F}_i(\widehat{\mu}) + g_i(\widehat{x})] = 0 \quad (i \in \overline{1,m}).$$

Let $\varepsilon > 0$ and $(x, \mu) \in \mathcal{D}$ be given and define $\alpha \in \mathbf{R}^{m+1}$ as follows

$$\alpha_0 := \mathcal{F}_0(\mu) + g_0(x) + \varepsilon,$$
$$\alpha_i := \mathcal{F}_i(\mu) + g_i(x) \quad (i \in \overline{1,m}).$$

Then /see the definition of C/ $(\alpha_0, \alpha_1, \dots, \alpha_m) \in C$, i.e.

$$\sum_{i=0}^{m} \widehat{\lambda}_i [\mathcal{F}_i(\mu) + g_i(x)] + \varepsilon \widehat{\lambda}_0 \geq 0.$$

If we take the function $0 < \varepsilon \mapsto \sum_{i=0}^{m} \widehat{\lambda}_i [\mathcal{F}_i(\mu) + g_i(x)] + \varepsilon \widehat{\lambda}_0$ and its limit at zero, then it follows that

$$\sum_{i=0}^{m} \widehat{\lambda}_i [\mathcal{F}_i(\mu) + g_i(x)] \geq 0. \tag{12}$$

Now let $(x,\mu) := (\widehat{x},\widehat{\mu})$ and take into account that $(\widehat{x},\widehat{\mu}) \in \mathcal{D}$. Then by (5) and (4)

$$\sum_{i=0}^{m} \widehat{\lambda}_i [\mathcal{F}_i(\widehat{\mu}) + g_i(\widehat{x})] = 0 \tag{13}$$

is true. Therefore the equalities

$$\min\{\sum_{i=0}^{m} \widehat{\lambda}_i [\mathcal{F}_i(\mu) + g_i(x)] \mid (x,\mu) \in \mathcal{D}\} =$$

$$= \sum_{i=0}^{m} \widehat{\lambda}_i [\mathcal{F}_i(\widehat{\mu}) + g_i(\widehat{x})] = 0$$

follow immediately from (13) and (14). Since the last equality is equivalent to (1) and (2), the necessity part of our theorem is proved.

ii/The proof of the sufficiency is very simple. Indeed, by the assumptions, there exist a number $\widehat{\lambda}_0 > 0$ and a pair $(\widehat{x},\widehat{\mu}) \in \mathcal{D}$ such that the equality

$$\min\{\sum_{i=0}^{m} \widehat{\lambda}_i [\mathcal{F}_i(\mu) + g_i(x)] \mid (x,\mu) \in \mathcal{D}\} = \sum_{i=0}^{m} \widehat{\lambda}_i [\mathcal{F}_i(\widehat{\mu}) + g_i(\widehat{x})]$$

holds. However, $\widehat{\lambda}_i \geq 0$ $(i \in \overline{1,m'})$ thus, by the definition of $\mathcal{D}$ and (4), we get that for $(x,\mu) \in \mathcal{D}$

$$\widehat{\lambda}_0 [\mathcal{F}_0(\mu) + g_0(x)] \geq \sum_{i=0}^{m'} \widehat{\lambda}_i \, [\mathcal{F}_i(\mu) + g_i(x)] =$$

$$= \sum_{i=0}^{m} \widehat{\lambda}_i \, [\mathcal{F}_i(\mu) + g_i(x)] \geq$$

$$\geq \sum_{i=0}^{m} \widehat{\lambda}_i \, [\mathcal{F}_i(\widehat{\mu}) + g_i(\widehat{x})] = \widehat{\lambda}_0 \, [\mathcal{F}_0(\widehat{\mu}) + g_0(\widehat{x})] \,.$$

This implies by $\widehat{\lambda}_0 > 0$ for all $(x, \mu) \in \mathcal{D}$ that

$$\mathcal{F}_0(\widehat{\mu}) + g_0(\widehat{x}) \leq \mathcal{F}_0(\mu) + g_0(x)$$

i.e. $(\widehat{x}, \widehat{\mu})$ is optimal and the proof of Theorem 3 is complete.

REMARK. Applying the above notations let us introduce the Lagrange function

$$\mathcal{L} : X_0 \times \mathcal{U} \times \mathbf{R} \times \mathbf{R}^m \to \mathbf{R},$$

$$\mathcal{L}(x, \mu, \lambda_0, \lambda) := \sum_{i=0}^{m} \widehat{\lambda}_i \left[\mathcal{F}_i(\mu) + g_i(x)\right].$$

By means of the Lagrange function the conditions (1) and (2) of Theorem 3 can be formulated in the following way

$$\min\{\mathcal{L}(x, \mu, \lambda_0, \lambda) \mid (x, \mu) \in X_0 \times \mathcal{U}\} = \mathcal{L}(\widehat{x}, \widehat{\mu}, \widehat{\lambda}_0, \widehat{\lambda}).$$

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S. E. GYARMATI

Dept. of Applied Analysys,
Eötvös Loránd University
H-1088 Budapest, Múzeum krt. 6-8

HUNGARY