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NOTES ON LEE'S HARMONIC FIT ALGORITHM

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Abstract. One of the best of the known on-line heuristics for the classical one-dimensional bin-packing problem is Harmonic Fit, which was developed by C.C. Lee and D.T. Lee [2]. They proved that the asymptotic worst-case performance ratio of this algorithm is 1.6901.... Using more general sequences we analyse Harmonic Fit for those types of lists, for which the maximum size of the elements is bounded by a reciprocal value of a given integer $r \geq 1$. As another generalization we give a parametric lower bound for $O(1)$-space on-line bin-packing algorithms.

Introduction

One of the most frequently studied NP-complete problems is the classical bin-packing problem: we have to pack into the minimum number of unit capacity bins a list of n elements $a_i, i = 1, 2, \ldots, n$, whose sizes are not greater than one. In the last decade numerous polinomial algorithms have been developed to give approximate solution for this problem. Some of these algorithms are on-line in the sense that they pack only one element at a time without knowing anything about the unpacked elements, and they do not move elements already packed. One possibility to measure the efficiency of an algorithm is to analyse its worst-case behaviour. This may be characterized by the asymptotic worst-case performance ratio, which can be defined in the following way. Let $A(L)$

- resp. L^* - the number of bins used by the algorithm A - resp. the optimal packing - and let $R_A(k)$ denote the supremum of the quotients of $A(L)/L^*$ for all lists with $L^* = k$. Then the asymptotic performance ratio (APR) is $R_A = \limsup_{k\to\infty} R_A(k)$. If $\frac{1}{r+1} < \max a_i \le \frac{1}{r}$ for some integer $r \ge 1$ we denote the APR by R_A^r (of course $R_A^1 = R_A$).

One of the best of the known on-line algorithms is the so-called Harmonic Fit (HF). This heuristic has been defined in the following way [3]: Let $(a_1, \ldots, a_n)$ be a list, $a_i \in (0,1], 1 \le i \le n$. We define $M(\ge 1)$ subintervals: $I_k = (\frac{1}{k+1}, \frac{1}{k}], 1 \le k < M$, and $I_M = (0, \frac{1}{M}]$. An item a_i is to be called an I_k-piece if $a_i \in I_k$. The bins are also classified into M categories: a bin designated to pack I_k-pieces exclusively is called an I_k-bin. The algorithm opens M empty bins at a time. If the subsequent element is an I_k-piece then the HF algorithm tries to pack it into the last opened I_k-bin. If there is not enough space then closes this bin, opens a new one and packs the element into the newly opened bin.

Starting from the $t_1 = 1$, $t_{i+1} = t_i(t_i + 1)$, $i \ge 1$, sequence C.C. Lee and D.T. Lee proved that $R_{HF} = 1.6901$ if $M \ge 5$. Using more general sequences which have been used in [1] and [2] as well, we shall prove that $R_{HF}^2 = 1.423\ldots$, $R_{HF}^3 = 1.302\ldots$, $R_{HF}^4 = 1.234\ldots$. Our result contains the special case $r = 1$, Lee's ratio.

Since in [2] it has been proved that the lowest achiavable values for any on-line algorithm A are $R_A^1 = 1.5364\ldots, R_A^2 = 1.365\ldots, R_A^3 = 1.274\ldots, R_A^4 = 1.219\ldots, \ldots$ so our result shows that the HF- algorithm belongs to the best ones in the parametric case as well. As an other simple generalization we prove that the parametric bounds for HF are sharp lower bounds for any 0(1)-space on-line algorithms.

Worst-case analysis of HF with parameter r.

For the sake of simplicity we use the terminology of [3]. The main difference is that in the sequel we suppose the size of the elements to be in the interval $(0,t]$, where $\frac{1}{r+1} < t \leq \frac{1}{r}$ for some positive integer r.

Let us divide the interval $(0,t]$ into subintervals: $(0,t] = \cup_{k=r}^{M} I_k$, where $M > 0$ integer, and $I_k = (\frac{1}{k+1}, \frac{1}{k}], r \leq k < M, I_M = (0, \frac{1}{M}]$. We denote the number of I_k-bins by m_k, the number of I_k-elements in the list by n_k. Let $s_M = \sum_{a_i \in I_M} a_i$. We denote by $HF_M^r(L)$ the number of bins used by the algorithm HF with parameters r and M. It is clear that

$$HF_M^r(L) = \sum_{k=r}^{M-1} \lceil \frac{n_k}{k} \rceil + \lceil \frac{M \cdot s_M}{M-1} \rceil < \\ < \sum_{k=r}^{M-1} \frac{n_k}{k} + \frac{M \cdot s_M}{M-1} + (M-r) \tag{2.1}$$

Let $g(x)$ the ratio-function defined in [1] and [3]:

$$g(x) = \begin{cases} \frac{1}{k}, & \text{if} \quad x \in I_k, r \leq k < M, \\ \frac{M \cdot x}{M-1}, & \text{if} \quad x \in I_M \end{cases}$$

The main idea of the analysis is that the HF algorithm is invariant for the permutations of the elements of L and so if the bin B_i in L^* contains the elements $a_{k_1}, \ldots, a_{k_i}$, then

$$HF_M^r(L) \leq \sum_{i=1}^{L^*} \sum_{a_{k_j} \in B_i} g(a_{k_j}) + (M-r) \tag{2.2}$$

Let $S_r = \{(y_1, \ldots, y_v), y_i > 0, \ 1 \leq i \leq v, \ \frac{1}{r+1} \max_{1 \leq i \leq v} y_i \leq \frac{1}{r}$ and $\sum_{i=1}^{v} y_i \leq 1\}$ be a vector set and let

$$\overline{G}_M^r = \sup_{S_r} \sum_{i=1}^{v} g(y_i).$$

So

$$HF_M^r(L) < \sum_{i=1}^{n} g(a_i) + (M-r) = \sum_{i=1}^{L^*} \sum_{a_j \in B_i} g(a_j) + (M-r)$$
$$\leq L^* \overline{G_M^r} + (M-r),$$

and so

$$R_{HF}^r = \limsup_{L^* \to \infty} \frac{HF_M^r(L)}{L^*} \leq \overline{G_M^r} \tag{2.3}$$

Let us define the following sequences for different $r > 0$ integers:

$$t_1(r) = r+1 \text{ and } t_{i+1}(r) = 1 + \prod_{j=1}^{i} t_j(r) \tag{2.4}$$

Since it does not make trouble, in the sequel we use the notation t_i instead of $t_i(r)$.

Theorem 1. *For $i > r$, $t_i < M+1 \leq t_{i+1}$, then*

$$\overline{G_M^r} = 1 + \sum_{j=2}^{i} \frac{1}{t_j - 1} + \frac{M}{(t_{i+1}-1)(M-1)}.$$

Proof. First let us suppose that $Y = (y_1, \ldots, y_v) \in S_r$ contains $r - 1 (= t_1 - 2)$ I_{t_1-1}-elements. If each of the remainder elements is in the intervals $I_{t_2-1}, I_{t_3-1}, \ldots, I_M$, and we allow more than one I_M-elements then

$$\sum_{y_j \in Y} g(y_j) = 1 + \sum_{j=2}^{i} \frac{1}{t_j - 1} + \frac{M}{M-1} \cdot \sum_{j=i+1}^{v} y_j \leq \overline{G_M^r} \tag{2.5}$$

We note that equality holds in (2.5) if $y_j = \frac{1}{t_j - 1} + \varepsilon$, $j = 1, \ldots, i$, and $\sum_{j=i+1}^{v} y_j = \frac{1}{t_{i+1}-1} - i\varepsilon$ for a sufficiently small $\epsilon > 0$.

Let us now suppose that, there exists an index $r-1 < l \le i$ for which $y_1 \notin I_{t_i - r+2}$. Since $\frac{1}{t_l - 1} = 1 - \sum_{i=1}^{l-1} \frac{1}{t_1 - 1}$, and

$$\sum_{j \geq 1} g(y_j) \le \frac{t_l + 1}{t_l} \cdot \frac{1}{t_l - 1} = \frac{1}{t_l - 1} \cdot \frac{1}{t_l(t_l - 1)},$$

So

$$\sum_{y_j \in Y} g(y_j) \le 1 + \sum_{j=2}^{l+1} \frac{1}{t_j - 1} \le 1 + \sum_{j=2}^{i+1} \frac{1}{t_j - 1} < \overline{G}^r_M .$$

If the number of the $I_{t_1 - 1}$-elements is $q \le r - 2 = t_1 - 3$, then $\sum_{i=1}^{q} g(y_i) = \frac{q}{t_1 - 1}$. We know that $y_j \le \frac{1}{t_1}$ if $j > q$, and so $\sum_{j>q} g(y_j) \le \frac{t_1+1}{t_1}(1 - \frac{q}{t_1})$. Therefore

$$\sum_{y_j \in Y} g(y_j) \le \frac{q}{t_1 - 1} + \frac{t_1 + 1}{t_1}(1 - \frac{q}{t_1}) \le \frac{t_1 - 3}{t_1 - 1} + \frac{3(t_1 - 1)}{t_1^2} \le$$

$$\le 1 + \sum_{j=2}^{3} \frac{1}{t_j - 1} < \overline{G}^r_M .$$

Because of the note below the inequality (2.5) the following corollary is true.

COROLLARY 1.

$$\lim_{M \to \infty} R_{HF^r_M} = \lim_{M \to \infty} \overline{G}^r_M = \sum_{j=1}^{\infty} \frac{1}{t_j - 1} = 1 + \sum_{j=2}^{\infty} \frac{1}{t_j - 1}.$$

Using the sequences (2.4) we can easily generalize the Theorem 3 of [3] in the following way:

Theorem 2. *Let A be any $0(1)$-space on-line bin-packing algorithm, and let $r > 0$ integer. Then*

$$R_A^r \geq 1 + \sum_{i=1}^{\infty} \frac{1}{t_i(r) - 1}.$$

Proof. Let us suppose that the list L contains $(t_1(r) - 1)n$ pieces of $a_1 = \frac{1}{t_1(r)} + \varepsilon$ elements, and n elements of each $a_j = \frac{1}{t_j(r)} + \varepsilon$, $j = 2, \ldots, i$, where $\varepsilon > 0$ is sufficiently small. If we suppose that the algorithm A can use only K "active" bins then it is easy to see that $A(L) \leq n\left(1 + \frac{1}{t_2(r)-1} + \ldots + \frac{1}{t_i(r)-1}\right) - (t_2(r) - 1) \cdot K$ and $L^* = n$. So

$$R_A^r \geq 1 + \sum_{j=2}^{i} \frac{1}{t_j(r) - 1} .$$

If i tends to infinity we get the statement of the theorem.

References

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