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ON DOMAIN OF ATTRACTION OF THE DOUBLE EXPONENTIAL DISTRIBUTION*

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1.Introduction

For a sequence of independent, identically distributed random variables $X_1, X_2, \ldots$ with common distribution function (d.f.) $F(x)$ and for $n \geq 1$, let

$$Z_n = \max(X_1, X_2, \ldots, X_n).$$

Then, obviously,

$$P(Z_n < x) = F^n(x).$$

We say that F belongs to the domain of attraction of a nondegenerate d.f. H, if there exist sequences of constants $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ with $b_n > 0$, such that

$$\lim_{n \to \infty} F^n(a_n + b_n x) = H(x) \tag{1}$$

holds at all continuity points of H. The relation (1) will be denoted by $F \in D(H)$.
The limit laws for Z_n were fully characterized in [4]. They are the so-called extreme value distributions. We employ the notation $H_{1,\gamma}(x) = \exp(-x^{-\gamma}), x > 0, H_{2,\gamma}(x) = \exp(-(-x)^{\gamma})$, $x < 0$

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and $H_{3,0}(x) = \exp(-e^{-x})$, $-\infty < x < +\infty$, where γ is a positive parameter. A d.f. F can belong only to the domain of attraction of one of the three types $H_{1,\gamma}, H_{2,\gamma}$ and $H_{3,0}$ (see [4] or [3], for example).

Necessary and sufficient conditions for $F \in D(H)$ are well-known (see [5] and references above). In a recent paper [2] we gave characterizations of domains of attraction of extreme value distributions in terms of ultimately convexity (concavity) properties of some functions $[1-G(x)]^q$ ($q \in \mathbf{R}\backslash\{0\}$), where G is an appropriate d.f. depending only on F. This approach seems to be new.

In [4] Gnedenko remarked that the characterization of $H_{3,0}$ (the double exponential distribution) cannot be regarded as final and simple enough. In this paper we show that each d.f. F in the domain of attraction of $H_{3,0}$ is asymptotic to some twice differentiable d.f. G, where $[1 - G(x)]^q$ is ultimately convex for every $q \in \mathbf{R}\backslash\{0\}$. This result is closely related to the Theorem of [1].

2. The result

For a given d.f. F, let

$$\omega(F) = \sup\{x; F(x) < 1\}.$$

We say that the function $[1 - F(x)]^q$ ($q \in \mathbf{R}\backslash\{0\}$ is fixed) is ultimately convex if for the given q there exists an $x(q) < \omega(F)$ such that $[1 - F(x)]^q$ is convex on the interval $(x(q), \omega(F))$.

Theorem. *$F \in D(H_{3,0})$ if and only if there exists a twice differentiable d.f. G such that*

a.) $\omega(G) = \omega(F) = \omega$

b.) $[1 - G(x)]^q$ *is ultimately convex for every* $q \in \mathbf{R}\backslash\{0\}$

c.) $\lim_{x\uparrow\omega} \frac{1-F(x)}{1-G(x)} = 1$. □

The proof will be based on the following lemmas.

Lemma 1. *For a given twice differentiable d.f. G,* $[1-G(x)]^q$

is ultimately convex for every $q \in \mathbf{R}\setminus\{0\}$, *if and only if*

$$\lim_{x \uparrow \omega} \left[\frac{1 - G(x)}{G'(x)} \right]' = 0,$$

where commas denote derivatives. □

Proof. The statement follows directly from the Proposition of [2]. □

The basic idea we use to construct the d.f. G is due to Balkema and de Haan, see [1]. Assume that $F \in D(H_{3,0})$. Let $\overline{F}_0(x) = 1 - F(x)$ and

$$F_{k+1}(x) = \max(0, 1 - \int_x^\omega \overline{F}_k(t)dt)$$

for $k = 0, 1, 2$ and $\overline{F}_k(x) = 1 - F_k(x)$. It follows from [5] that the integrals in the definition of $F_{k+1}(x)$ are finite and $F_{k+1}(x)$ is a d.f. also for $k = 0, 1, 2$ with $\omega(F_{k+1}) = \omega(F) = \omega$. In [5] it was proved also that if $F \in D(H_{3,0})$ then $F_{k+1} \in D(H_{3,0})(k = 0, 1, 2)$ and the following relation is valid:

$$\lim_{x \uparrow \omega} \frac{\overline{F}_k(x) \cdot \overline{F}_{k+2}(x)}{\{\overline{F}_{k+1}(x)\}^2} = 1, \; k = 0, 1. \tag{2}$$

Let us define the function $A(x)$ by

$$A(x) = \frac{\{\overline{F}_2(x)\}^3}{\{\overline{F}_3(x)\}^2} \quad \text{for } x \in (x_0, \omega), \tag{3}$$

where $x_0 < \omega$ is such that $\overline{F}_3(x_0) > 0$.

Lemma 2. *If* $F \in D(H_{3,0})$ *then there exists an* x_1 *such that* $x_1 \in (x_0, \omega) and A(x)$ *is strictly decreasing on the interval* (x_1, ω). □

Proof. It is obvious that

$$A'(x) = A(x) \cdot \frac{\overline{F}_2(x)}{\overline{F}_3(x)} \cdot \left[-3 \cdot \frac{\overline{F}_1(x) \cdot \overline{F}_3(x)}{\{\overline{F}_2(x)\}^2} + 2\right]. \tag{4}$$

By relation (2) it follows that there exists an $x_1 \in (x_0, \omega)$ such that $A'(x) < 0$ for $x \in (x_1, \omega)$. □

Lemma 3. *If $F \in D(H_{3,0})$then$\{A(x)\}^q$ is ultimately convex for every $q \in R\backslash\{0\}$.* □

Proof. It will be sufficient to show that

$$\lim_{x \uparrow \omega} \left[\frac{A(x)}{A'(x)}\right]' = 0.$$

From (4) we get

$$\frac{A(x)}{A'(x)} = \frac{\overline{F}_3(x)[\overline{F}_2(x)]^{-1}}{2 - 3 \cdot \overline{F}_1(x) \cdot \overline{F}_3(x) \cdot [\overline{F}_2(x)]^{-2}}, \text{ whence}$$

$$\left[\frac{A(x)}{A'(x)}\right]' = \frac{-1 + \overline{F}_3(x) \cdot \overline{F}_1(x) \cdot [\overline{F}_2(x)]^{-2}}{2 - 3 \cdot \overline{F}_1(x) \cdot \overline{F}_3(x) \cdot [\overline{F}_2(x)]^{-2}} -$$

$$-3 \cdot \frac{-\overline{F}_0(x) \cdot [\overline{F}_3(x)]^2 \cdot [\overline{F}_2(x)]^{-3} - \overline{F}_1(x) \cdot \overline{F}_3(x) \cdot [\overline{F}_2(x)]^{-2}}{\{2 - 3 \cdot \overline{F}_1(x) \cdot \overline{F}_3(x) \cdot [\overline{F}_2(x)]^{-2}\}^2}$$

$$+ \frac{2 \cdot [\overline{F}_1(x)]^2 \cdot [\overline{F}_3(x)]^2 \cdot [\overline{F}_2(x)]^{-4}}{\{2 - 3 \cdot \overline{F}_1(x) \cdot \overline{F}_3(x) \cdot [\overline{F}_2(x)]^{-2}\}^2} .$$

By relation (2) the denominators tend to $-1, 1, 1$, respectively, and the numerators tend to $0, -2, 2$, respectively. This completes the proof. □

Lemma 4. *Assume that $F \in D(H_{3,0})$ .Then*

$$\lim_{x \uparrow \omega} \frac{1 - F(x)}{A(x)} = 1. \tag{5}$$

Proof. By (3) we have that

$$\frac{1 - F(x)}{A(x)} = \frac{\overline{F}_0(x) \cdot \overline{F}_2(x)}{[\overline{F}_1(x)]^2} \cdot \left[\frac{\overline{F}_1(x) \cdot \overline{F}_3(x)}{[\overline{F}_2(x)]^2} \right]^2 .$$

So by (2) we get the relation (5). □
COROLLARY. If $F \in D(H_{3,0})$ then $\lim_{x \uparrow \omega} A(x) = 0$. □

Proof. The statement is evident from (5). □

Let us turn to the proof of our Theorem. The "if" part is a consequence of Theorem 2.7.2. of [3], Lemma 2.5 of [6] and our Lemma 1.

Assume now that $F \in D(H_{3,0})$ and let

$$G(x) = \begin{cases} \max(0, 1 - A(x)) & \text{for } x > x_1 \\ 0 & \text{for } x \leq x_1 . \end{cases} \tag{6}$$

$G(x)$ is a d.f. because of Lemma 2 and the Corollary. It is obvious by (3) and (6) that $\omega(G) = \omega(F)$. The part b.) follows from Lemma 3 and Lemma 1. The part c.) is a direct consequence of Lemma 4. This completes the proof. □

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