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IMPROVED ERROR OF AN ARBITRARY ORDER FOR THE APPROXIMATE SOLUTION OF SYSTEM OF ORDINARY DIFFERENTIAL EQUATIONS WITH SPLINE FUNCTIONS

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Abstract: This paper is concerned with the system of nonlinear ordinary differential equations $y' = f_1(x,y,z)$, $z' = f_2(x,y,z)$ with $y(x_0) = y_0$ and $z(x_0) = z_0$ where $f_1, f_2 \in C^r, r \in I^+$. An appropriate method for obtaining spline function approximations for the solution of this system is established. These spline functions are not necesseraly polynomial splines. It is shown that the method is a one-step method $O(h^{\alpha+r+m})$ in $y^{(q)}(x)$ and $z^{(q)}(x), q = o(1)r + 1$ where $0 < \alpha \leq 1$. Here m is an arbitrary positive integer which in fact equals the number of iteration processes describing the spline functions defined in the method.

Description of the Method

Consider the system of ordinary differential equations

$$y' = f_1(x,y,z), \qquad y(x_0) = y_0 \tag{1}$$
$$z' = f_2(x,y,z), \qquad z(x_0) = z_0, \tag{2}$$

where $f_1, f_2 \in C^r([0,1] \times R^2)$ and $r \in I^+$.

Let $\triangle$ be the partition

$$\triangle : 0 = x_0 < x_1 \ldots < x_k < x_{k+1} < \ldots < x_n = 1$$

where $x_{k+1} - x_k = h < 1$ and $k = O(1)n - 1$.

Let L_1 and L_2 be the Lipschitz constants satesfied by the functions $f_1^{(q)}$ and $f_2^{(q)}$ respectively, i.e.,

$$|f_1^{(q)}(x, y_1, z_1) - f_1^{(q)}(x, y_2, z_2)| \leq L_1\{|y_1 - y_2| + |z_1 - z_2|\}, \quad (3)$$

$$|f_2^{(q)}(x, y_1, z_1) - f_2^{(q)}(x, y_2, z)| \leq L_2\{|y_1 - y_2| + |z_1 - z_2|\} \quad (4)$$

for all (x, y, z_1) and (x, y_2, z_2) in the domain of definition of f_1 and f_2 and all $q = o(1)r$.

It should be noted that we use the Lipschitz conditions on f_1 and f_2 to guarantee the existence of unique solution, $y(x)$ and $z(x)$, to the problem (1)-(2).

The functions $f_1^{(q)}$ and $f_2^{(q)}$, $q = o(1)r$ are functions of x, y and z only and they are given from the following algorithm :

Let $f_1^{(0)} = f_1(x, y, z)$ and $f_2^{(0)} = f_2(x, y, z)$

Then, for all $q = 1(1)r$

$$y^{(q+1)} = \frac{d^q f_1}{dx^q} = f_1^{(q)} = \frac{\partial f_1^{(q-1)}}{\partial x} + \frac{\partial f_1^{(q-1)}}{\partial y} f_1 + \frac{\partial f_1^{(q-1)}}{\partial z} f_2$$

and

$$z^{(q+1)} = \frac{d^q f_2}{dx^q} = f_2^{(q)} = \frac{\partial f_2^{(q-1)}}{\partial x} + \frac{\partial f_2^{(q-1)}}{\partial y} f_1 + \frac{\partial f_2^{(q-1)}}{\partial z} f_2$$

Now, choosing the arbitrary positive intger m, we define the spline functions approximating the solution $y(x)$ and $z(x)$ by $S_\triangle$

and $\bar{S}_\Delta(x)$ where

$$S_\Delta(x) \equiv S_k^{[m]}(x) = S_{k-1}^{[m]}(x_k) + \int_{x_k}^{x} f_1[t_1, S_k^{[m-1]}(t_1), \bar{S}_k^{[m-1]}(t_1)]dt_1 \tag{5}$$

and

$$\bar{S}_\Delta(x) \equiv \bar{S}_k^{[m]}(x) = \bar{S}_{k-1}^{[m]}(x_k) + \int_{x_k}^{x} f_2[t_1, S_k^{[m-1]}(t_1), \bar{S}_k^{[m-1]}(t_1)]dt_1 \tag{6}$$

where

$x_k \le x \le x_{k+1}$, $k = o(1)n-1$, $S_{-1}^{[m]}(x_0) = y_0$ and $\bar{S}_{-1}^{[m]}(x_0) = z_0$.

To equations (5) and (6) we associate the following m iteration processes:

For $x_k \le x \le x_{k+1}$, $k = o(1)n-1$,

$$S_k^{[0]}(x) = S_{k-1}^{[m]}(x_k) + \sum_{j=0}^{r} \frac{(x-x_k)^{j+1}}{(j+1)!} \cdot f_1^{(j)}\{x_k, S_{k-1}^{[m]}(x_k), \bar{S}_{k-1}^{[m]}(x_k)\}, \tag{7}$$

$$\bar{S}_k^{(0)}(x) = \bar{S}_{k-1}^{[m]}(x_k) + \sum_{j=0}^{r} \frac{(x-x_k)^{j+1}}{(j+1)!} \cdot f_2^{(j)}\{x_k, S_{k-1}^{[m]}(x_k), \bar{S}_{k-1}^{[m]}(x_k)\},$$

$$S_k^{[j]}(x) = S_{k-1}^{[m]}(x_k) + \int_{x_k}^{x} f_1[t_{m-j+1}, S_k^{[j-1]}(t_{m-j+1}),$$

$$, \bar{S}_k^{[j-1]}(t_{m-j+1})]dt_{m-j+1}$$

$$\bar{S}_k[j](x) = \bar{S}_{k-1}^{[m]}(x_k) + \int_{x_k}^{x} f_2[t_{m-j+1}, S_k^{[j-1]}(t_{m-j+1}),$$

$$, \bar{S}_k^{[j-1]}(t_{m-j+1})]dt_{m-j+1}$$

where $j = 1(1)m$.

By construction, it is obvious that $S_\Delta(x)$ and $\bar{S}_\Delta(x) \in C[0,1]$.

Error Estimation and Convergence

The exact solution to the problem (1) - (2) can be written in the following forms:

For all $x_k \leq x \leq x_{k+1}$, $\quad k = 0(1)n-1$,

$$y(x) \equiv y^{[m]}(x) = y_k + \int_{x_k}^{x} f_1[t_1, y^{[m-1]}(t_1), z^{[m-1]}(t_1)]dt_1 \qquad (8)$$

and

$$z(x) \equiv z^{[m]}(x) = z_k + \int_{x_k}^{x} f_2[t_1, y^{[m-1]}(t_1), z^{[m-1]}(t_1)]dt_1 \qquad (9)$$

where the following m-iteration processess are considered,

$$y^{[0]}(x) = y_k + \sum_{j=1}^{r} \frac{(x-x_k)^j}{j!} y_k^{(j)} + \qquad (10)$$

$$+ \frac{y^{(r+1)}(\xi_k)}{(r+1)!}(x-x_k)^{r+1}$$

$$z^{[0]} = (x) = z_k + \sum_{j=1}^{r+1} \frac{(x-x_k)^j}{j!} z_k^{(j)} +$$

$$+ \frac{z^{(r+1)}(\eta_k)}{(r+1)!}(x - x_k)^{r+1}$$

$$y^{[j]}(x) = y_k + \int_{x_k}^{x} f_1[t_{m-j+1}, y^{[j-1]}(t_{m-j+1}),$$

$$, z^{[j-1]}(t_{m-j+1})]dt_{m-j+1}$$

$$z^{[j]}(x) = z_k + \int_{x_k}^{x} f_2[t_{m-j+1}, y^{[j-1]}(t_{m-j+1}),$$

$$, z^{[j-1]}(t_{m-j+1})]dt_{m-j+1}$$

where $j = 1(1)m$ and $\xi_k, \eta_k \in (x_k, x_{k+1}), k = o(1)n - 1$.

Here $x_k \leq t_m \leq t_{m-1} \leq \ldots \leq t_{m-j+1} \leq \ldots \leq t_1 \leq x \leq x_{k+1}$.

First, we consider the subinterval $I_0 = [x_0, x_1]$.

To estimate $|y(x) - S_0^{[m]}(x)|$, we use both (5) and (8) for $k = 0$, both (7) and (10) for $k = 0$ and the Lipschitz conditions (3)-(4), so we get:

$$|y(x) - S_0^{[m]}(x)| \leq L_1 \int_{x_0}^{x} \{|y^{[m-1]}(t_1) - S_0^{[m-1]}(t_1)| + |z^{[m-1]}(t_1) -$$

$$- \bar{S}_0^{[m-1]}(t_1)\}dt_1,$$

$$\leq L_1^2 \int_{x_0}^{x} \int_{x_0}^{t_1} \{|y^{[m-2]}(t_2) - S_0^{[m-2]}(t_2)| +$$

$$+ |z^{m-2}(t_2) - \bar{S}_0^{[m-2]}(t_2)|\}dt_2 dt_1 +$$

$$+ L_1 L_2 \int_{x_0}^{x} \int_{x_0}^{t_1} \{|y^{[m-2]}(t2) - S_0^{[m-2]}(t_2)| +$$

$$+ |z^{m-2}(t_2) - \bar{S}_0^{[m-2]}(t_2)|\} dt_2 dt_1$$

$$\leq L_1^3 \int_{x_0}^{x} \int_{x_0}^{t_1} \int_{x_0}^{t_2} \{|y^{[m-3]}(t_3) - S_0^{[m-3]}(t_3)| +$$

$$+ |z^{m-3}(t_3) - \bar{S}_0^{[m-3]}(t_3)|\} dt_3 dt_2 dt_1 +$$

$$+ 2L_1^2 L_2 \int_{x_0}^{x} \int_{x_0}^{t_1} \int_{x_0}^{t_2} \{|y^{[m-3]}(t3) - S_0^{[m-3]}(t_3)| +$$

$$+ |z^{m-3}(t_3) - \bar{S}_0^{[m-3]}(t_3)|\} dt_3 dt_2 dt_1 +$$

$$+ L_1 L_2^2 \int_{x_0}^{x} \int_{x_0}^{t_1} \int_{x_0}^{t_2} \{|y^{[m-3]}(t3) - S_0^{[m-3]}(t_3)| +$$

$$+ |z^{m-3}(t_3) - \bar{S}_0^{[m-3]}(t_3)|\} dt_3 dt_2 dt_1 +$$

$$\leq \dots$$

$$\leq \sum_{j=0}^{m-1} (m-1) C_j L_1^{m-j} L_2^j \int_{x_0}^{x} \int_{x_0}^{t_1} \dots \int_{x_0}^{t_{m-1}} \{|y^{[0]}(t_m) -$$

$$- S_0^{[0]}(t_m)| + |z^{[0]}(t_m) -$$

$$\bar{S}_0^{[0]}(t_m)|\} dt_m dt_{m-1} \dots dt_2 dt_1$$

Using (7) and (10), it easy to get:

$$|y(x) - S_\Delta(x)| \leq \frac{2c_0}{(r+m+1)!} h^{r+m+1} \cdot \omega(h) = O(h^{\alpha+r+m+1}) \tag{11}$$

where

$c_0 = \sum_{j=0}^{m-1} {}^{m-1}c_j L_1^{m-j} L_2^j$, $\omega(h) = \max\{\omega(y^{(r+1)}, h), \omega(z^{(r+1)}, h)\}$

and $\omega(y^{(r+1)},h)$, $\omega(z^{(r+1)},h)$ are the moduli of continuity of the functions $y^{(r+1)}(x)$ and $z^{(r+1)}(x)$ respectively.

Similarly, using both (6) and (9) for $k=0$ and both (7) and (10) for $k=0$ together with the Lipschitz conditions (3)-(4), it can be easily shown that:

$$|z(x)-\bar{S}_\Delta(x)| \le \frac{2c_1}{(r+m+1)!}h^{r+m+1}\cdot \omega(h) = O(h^{\alpha+r+m+1}) \tag{12}$$

where $c_1 = \sum_{j=0}^{m-1} {}^{m-1}C_j L_1^j L_2^{m-j}$.

We now estimate $|y'(x)-S'_\Delta(x)|$.

Using both (5) and (6) for $k=0$, both (7) and (10) for $k=0$ and the Lipschitz conditions (3)-(4), we get:

$$\begin{aligned}
|y'(x)-S'_\Delta(x)| &\le L_1\{|y^{[m-1]}(x)-S_0^{[m-1]}(x)|+ \\
&\quad + |z^{[m-1]}(x)-\bar{S}_0^{[m-1]}(x)|\} \\
&\le L_1^2\int_{x_0}^{x}\{|y^{[m-2]}(t_2)-S_0^{[m-2]}(t_2)|+ \\
&\quad + |z^{[m-2]}(t_2)-\bar{S}_0^{[m-2]}(t_2)|\}dt_2+ \\
&\quad + L_1L_2\int_{x_0}^{x}\{|y^{[m-2]}(t_2)-S_0^{[m-2]}(t_2)|+ \\
&\quad + |z^{[m-2]}(t_2)-\bar{S}_0^{[m-2]}(t_2)|\}dt_2 \\
&\le \dots \\
&\le c_0\int_{x_0}^{x}\int_{x_0}^{t_2}\int_{x_0}^{t_3}\dots\int_{x_0}^{t_{m-1}}\{|y^{[0]}(t_m)|-S_0^{[0]}(t_m)|+ \\
&\quad + |z^{[0]}(t_m)-\bar{S}_0^{[0]}(t_m)|\}dt_m\,dt_{m-1}\dots dt_3\,dt_2
\end{aligned}$$

Using (7) and (10) , we get:

$$|y'(x) - S'_\triangle(x)| \le \frac{2c_0}{(r+m)!} h^{r+m} \omega(h) = O(h^{\alpha+r+m}) \qquad (13)$$

In a similar manner, using both (6) and (9) for $k = 0$ and both (7) and (10) for $k = 0$ together with the Lipschitz conditions (3)-(4), we can show that :

$$|z'(x) - \bar{S}'_\triangle(x)| \le \frac{2c_1}{(r+m)!} h^{r+m} \omega(h) = O(h^{\alpha+r+m}) \qquad (14)$$

From (3), and (4), (13) and (14), it can be shown that:

$$y^{(q)}(x) - S^{(q)}_\triangle(x)| \le \frac{2C_0}{(r+m)!} h^{r+m} \omega(h) = O(h^{\alpha+r+m}) \qquad (15)$$

and

$$z^{(q)}(x) - \bar{S}^{(q)}_\triangle(x)| \le \frac{2C_1}{(r+m)!} h^{r+m} \omega(h) = O(h^{\alpha+r+m}) \qquad (16)$$

where $q = 2(1)r + 1$.

Then, we consider the general subinterval $I_k = [x_k, x_{k+1}]$, $k = 1(1)n - 1$.

To estimate $|y(x) - S_\triangle(x)|$, we give the following notations:

$$\begin{aligned} e(x) &= |y(x) - S - \triangle(x)|, \\ \bar{e}(x) &= |z(x) - \bar{S}_\triangle(x)|, \\ e_k &= |y_k - S_\triangle(x_k)| \\ \bar{e}_k &= |z_k - \bar{S}_\triangle(x)| \end{aligned}$$

Then, using (5), (7), (8), (10) and the Lipschitz conditions .(3)-(4), we get:

$$e(x) \le e_k + L_1 \int_{x_k}^{x} \{|y^{[m-1]}(t_1 - S_k^{[m-1]}(t_1)| + |z^{[m-1]}(t_1) -$$

$$- \bar{S}_k^{[m-1]}(t_1))|\}dt_1$$

$$\leq e_k(1 + L_1 h) + L_1 h\bar{e}_k + L_1^2 \int_{x_k}^{x}\int_{x_k}^{t_1} \{|y^{[m-2]}(t_2) - S_k^{[m-2]}(t_2)|+$$

$$+ |z^{[m-2]}(t_2) - \bar{S}_k[m-2](t_2)|\}dt_2 dt_1 +$$

$$+ L_1 L_2 \int_{x_k}^{x}\int_{x_k}^{t_1} \{|y^{[m-2]}(t_2) - S_k^{[m-2]}(t_2)|+$$

$$+ |z^{[m-2]}(t_2) - \bar{S}_k^{[m-2]}(t_2)|\}dt_2 dt_1$$

$$\leq e_k(1 + L_1 h + L_1^2 \frac{h^2}{2} + L_1 L_2 \frac{h^2}{2})+$$

$$+ \bar{e}_k(L_1 h + L_1^2 \frac{h^2}{2} + L_1 L_2 \frac{h^2}{2})+$$

$$+ L_1^3 \int_{x_k}^{x}\int_{x_k}^{t_1}\int_{x_k}^{t_2} \{|y^{[m-3]}(t_3) - S_k^{[m-3]}(t_3)|+$$

$$+ |z^{[m-3]}(t_3) - \bar{S}_k^{[m-3]}(t_3)|\}dt_3 dt_2 dt_1 +$$

$$+ 2L_1^2 L_2 \int_{x_k}^{x}\int_{x_k}^{t_1}\int_{x_k}^{t_2} \{|y^{[m-3]}(t_3) - S_k^{[m-3]}(t_3)|+$$

$$+ |z^{[m-3]}(t_3) - \bar{S}_k^{[m-3]}(t_3)|\}dt_3 dt_2 dt_1 +$$

$$+ L_1 L_2^2 \int_{x_k}^{x}\int_{x_k}^{t_1}\int_{x_k}^{t_2} \{|y^{[m-3]}(t_3) - S_k^{[m-3]}(t_3)|+$$

$$+ |z^{[m-3]}(t_3) - \bar{S}_k^{[m-3]}(t_3)|\}dt_3 dt_2 dt_1$$

$$\leq \ldots$$

$$\leq (1 + c_2 h)e_k + c_2 h\bar{e}_k +$$

$$+ c_0 \int_{x_k}^{x} \int_{x_k}^{t_1} \dots \int_{x_k}^{t_{m-1}} \{|y^{[0]}(t_m) - S_k^{[0]}(t_m)| +$$

$$+ |z^{[0]}(t_m) - \bar{S}_k^{[0]}(t_m)|\} dt_m \, dt_{m-1} \dots dt_2 \, dt_1$$

where c_2 is a constant independent of h.

Using (7) and (10), then for some constant c_3 independent of h, we get :

$$e(x) \le (1 + c_3 h) e_k + c_3 h \bar{e}_k + \frac{2c_0}{(r+m+1)!} h^{r+m+1} \omega(h) \qquad (17)$$

Similary, using (6), (7), (9), (10) and the Lipschitz conditions (3)-(4), it can be shown for some contant c_4 independent of h that:

$$\bar{e}(x) \le c_4 h e_k + (1 + c_4 h) \bar{e}_k + \frac{2c_1}{(r+m+1)!} h^{r+m+1} \omega(h) \qquad (18)$$

To achieve the proof of the convergence, we state the following definition of matrix inequality.

DEFINITION : Let $A = [a_{ij}], B = [b_{ij}]$ be two matrices of the same order, then we say that $A \le B$ iff

(i) a_{ij} and b_{ij} are non-negative,

(ii) $a_{ij} \le b_{ij} \forall i, j$.

Accordingly, if we use the matrix notations

$$E(x) = (e(x) \quad \bar{e}(x))^T$$
$$E_k = (e_k \quad \bar{e}_k)^T, \quad k = o(1)n - 1$$

.we can write the estimations (17) and (18) in the form :

$$E(x) \le \begin{bmatrix} 1 + c_3 h & c_3 h \\ c_4 h & 1 + c_4 h \end{bmatrix} E_k + \frac{2}{(r+m+1)!} h^{r+m+1} \omega(h) \begin{bmatrix} c_0 \\ c_1 \end{bmatrix}$$

or

$$E(x) \leq (I + hA)E_k + \frac{2}{(r+m+1)!} h^{r+m+1} \omega(h) B \tag{19}$$

where $A = \begin{bmatrix} c_3 & c_3 \\ c_4 & c_4 \end{bmatrix}$, $B = \begin{bmatrix} c_0 \\ c_1 \end{bmatrix}$ and I is the identity matrix of order 2.

Then, we give the following definition of the matrix norm: Let $T = [\tau_{ij}]$ be an $m \times n$ matrix, then we define

$$\|T\| = \max_i \sum_{j=1}^{n} |\tau_{ij}|$$

Using this definition, we get:

$$\|E(x)\| = \max(e(x) \quad \bar{e}(x)) \tag{20}$$

Since (19) is valid for all $x \in [x_k, x_{k+1}]$, $k = o(1)n-1$, then the following inequalities hold :

$$\|E(x)\| \leq (1 + h\|A\|)\|E_k\| + \frac{2}{(r+m+1)!} h^{r+m+1} \cdot \omega(h)\|B\|$$

$$(1 + h\|A\|)\|E_k\| \leq (1 + h\|A\|)^2 \|E_{k-1}\| + \frac{2}{(r+m+1)!} \cdot h^{r+m+1} \omega(h)\|B\|(1 + h\|A\|)$$

$$(1 + h\|A\|)^2 \|E_{k-1}\| \leq (1 + h\|A\|)^3 \|E_{k-2}\| + \frac{2}{(r+m+1)!} \cdot h^{r+m+1} \omega(h)\|B\|(1 + h\|A\|)^2$$

$$(1 + h\|A\|)^k \|E_1\| \leq (1 + h\|A\|)^{k+1} \|E_0\| + \frac{2}{(r+m+1)!} \cdot h^{r+m+1} \omega(h)\|B\|(1 + h\|A\|)^k$$

Adding L.H.S and R.H.S of these inequalities and noting that $\|E_0\| = 0$, we get :

$$\|E(x)\| \le \frac{2}{(r+m+1)!} h^{r+m+1}\omega(h)\|B\| \sum_{j=0}^{k}(1+h\|A\|)^j$$
$$\le c_5 h^{r+m}\omega(h)$$

where $c_5 = \frac{2\|B\|(e^{\|A\|}-1)}{(r+m+1)!\|A\|}$, is a constant independent of h.

Thus, using (20), we get:

$$e(x) \le c_5 h^{r+m}\omega(h) = O(h^{\alpha+r+m}) \tag{21}$$

and

$$\bar{e}(x) \le c_5 h^{r+m}\omega(h) = O(h^{\alpha+r+m}) \tag{22}$$

We now estimate $|y'(x) - S'_\Delta(x)|$. Using (5), (7), (8), (10) and the Lipschitz conditions (3)-(4), we get:

$$e'(x) \equiv |y'(x) - S'_\Delta(x)| \le L_1\{|y^{[m-1]}(x) - S_k^{[m-1]} + (x)| +$$
$$+ |z^{[m-1]}(x) - \bar{S}_k^{[m-1]}(x)|\}$$
$$\le L_1(e_k + \bar{e}_k) + L_1^2 \int_{x_k}^{x} \{|y^{[m-2]}(t_2) - S_k^{[m-2]}(t_2)| +$$
$$+ |z^{[m-2]}(t_2) - \bar{S}_k^{[m-2]}(t_2)|\}dt_2 + L_1 L_2 \int_{x_k}^{x} \{|y^{[m-2]}(t_2) -$$
$$- S_k^{[m-2]}(t_2)| + |z^{[m-2]}(t_2) - \bar{S}_k^{[m-2]}(t_2)|\}dt_2$$
$$\le e_k(L_1 + L_1^2 h + L_1 L_2 h) + \bar{e}_k(L_1 + L_1^2 + L_1 L_2 h) +$$
$$+ L_1^3 \int_{x_k}^{x} \int_{x_k}^{t_2} \{|y^{[m-3]}(t_3) - S_k^{[m-3]}(t_3)| + |z^{[m-3]}(t_3) -$$

$$- \bar{S}_k^{[m-3]}(t_3)|\}dt_3\,dt_2 +$$

$$+ 2L_1^2 L_2 \int_{x_k}^{x}\int_{x_k}^{t_2} \{|y^{[m-3]}(t_3) - S_k^{[m-3]}(t_3)| + |z^{[m-3]}(t_3) -$$

$$- \bar{S}_k^{[m-3]}(t_3)|\}dt_3\,dt_2 +$$

$$+ L_1 L_2^2 \int_{x_k}^{x}\int_{x_k}^{t_2} \{|y^{[m-3]}(t_3) - S_k^{[m-3]}(t_3)| + |z^{[m-3]}(t_3) -$$

$$- \bar{S}_k^{[m-3]}(t_3)|\}dt_3\,dt_2$$

$$\leq \dots$$

$$\leq c_6(e_k + \bar{e}_k) + c_0 \int_{x_k}^{x}\int_{x_k}^{t_2} \dots \int_{x_k}^{t_{m-1}} \{|y^{[0]}(t_m) -$$

$$- S_k^{[0]}(t_m)| + |z^{[0]}(t_m) - \bar{S}_k^{[0]}(t_m)|\}dt_m\,dt_{m-1}\dots dt_3\,dt_2$$

Then, for some constant c_7 independent of h and by using (7), (10), (21) and (22), we get:

$$e'(x) \leq 2c_5 c_7 h^{r+m}\omega(h) + \frac{2c_0}{(r+m)!}h^{r+m}\omega(h)$$

Thus, we have proved that:

$$e'(x) \leq c_8 h^{r+m}\omega(h) = O(h^{\alpha+r+m}) \tag{23}$$

where $c_8 = 2c_5c_7 + \frac{2c_0}{(r+m)!}$, is a constant independent of h.

In a similar manner, by using (6), (7), (9), (10), (21), (22) and the Lipschitz conditions (3)-(4), it can be shown that :

$$\bar{e}'(x) \equiv |z'(x) - \bar{S}'_\triangle(x)| \leq c_9 h^{r+m}\ \omega(h) = O(h^{\alpha+r+m}) \tag{24}$$

where c_9 is some constatnt identependent of h.

From (3), (4), (23) and (24), it can be shown that:

$$|y^{(q)}(x) - S_{\triangle}^{(q)}(x)| \leq c_8 h^{r+m} \ \omega(h) = O(h^{\alpha+r+m}) \qquad (25)$$

and

$$|z^{(q)}(x) - \bar{S}_{\triangle}^{(q)}(x)| \leq c_9 h^{r+m} \ \omega(h) = O(h^{\alpha+r+m}) \qquad (26)$$

where $q = 2(1)r + 1$.

Thus, we have proved the following theorem:

Theorem. *Let $y(x)$ and $z(x)$ be the exact solutions to the problem (1)-(2).If $S_{\triangle}(x)$ and $\bar{S}_{\triangle}(x)$, given in (5) and (6) are the approximate solutions, then the inequalities*

$$|y(x) - S_{\triangle}(x)| \leq \frac{2c_0}{(r+m+1)!} h^{r+m+1}\omega(h),$$

$$|z(x) - \bar{S}_{\triangle}(x)| \leq \frac{2c_1}{(r+m+1)!} h^{r+m+1}\omega(h),$$

$$|y(q)(x) - S_{\triangle}^{(q)}(x)| \leq \frac{2c_0}{(r+m)!} h^{r+m}\omega(h),$$

$$|z(q)(x) - \bar{S}_{\triangle}^{(q)}(x)| \leq \frac{2c_1}{(r+m)!} h^{r+m}\omega(h),$$

hold true for all $x \in [x_0, x_1]$ and all $q = 1(1)r + 1$ where

$$c_0 = \sum_{j=0}^{m-1} {}^{m-1}c_j L_1^{m-j} L_2^j \text{ and } c_1 = \sum_{j=0}^{m-1} {}^{m-1}c_j L_1^j L_2^{m-j}$$

and the inequalities

$$|y(q)(x) - S_{\triangle}^{(q)}(x)| \leq Ch^{r+m}\omega(h)$$

and

$$|z(q)(x) - \bar{S}_{\triangle}^{(q)}(x)| \leq Kh^{r+m}\omega(h)$$

hold true for all $x \in [x_k, x_{k+1}]$, $k = 1(1)n-1$ *and all* $q = o(1)r+1$ *where* C *and* K *are constants and independent of* h.

Numerical example:

Consider the following system of differential equations:

$$y' = y + z - x - x^2 - e^{2x}$$
$$z' = 2y + 2z - 2e^x - 2x^2 - 2, \qquad y(0) = 1, \quad z(0) = 2$$

The method is tested using this example in the interval $[0, 1]$ with step size $h = 0.1$ in the following cases :

(i) $r = 0$ and $m = 1, 2, 3$

(ii) $r = 1$ and $m = 1, 2$

The analytical solution is:

$$y(x) = e^x + x \text{ and } z(x) = e^{2x} + x^2 + 1$$

The tabulated resulted are evaluated at the point $x = 0.25$

(i) The case $r = 0$.

	Analytical value		Numerical value	Absolute error
y	1.534025417	m=1	1.530346203	3.679214E-03
		m=2	1.533757995	2.67422 E-04
		m=3	1.53400966	1.5757 E-05
y'	2.284025417	m=1	2.261907185	2.2118232E-02
		m=2	2.282660605	1.364812E-03
		m=3	2.283956955	6.8462 E-05
z	2.711221271	m=1	2.70386284	7.358431 E-03
		m=2	2.710686426	5.34845 E-04
		m=3	2.71118976	3.1511 E-05
z'	3.797442543	m=1	3.753206079	4.4236464E-02
		m=2	3.794712919	2.729624 E-03
		m=3	3.797305619	1.36924 E-04

(ii) The case $r = 1$.

y	1.534025417	m=1	1.533906117	1.193 E-04
		m=2	1.534018405	7.012 E-06
y'	2.284025417	m=1	2.283397416	6.28001 E-04
		m=2	2.283994225	3.1104 E-05
y''	1.284025155	m=1	1.282951722	1.073433 E-03
		m=2	1.283931843	9.3312 E-05
z	2.711221271	m=1	2.710982672	2.38599 E-04
		m=2	2.711207252	1.4019 E-05
z'	3.797442543	m=1	3.796186541	1.256002 E-03
		m=2	3.797380159	6.2384 E-05
z''	8.594884559	m=1	8.59273769	2.146869 E-03
		m=2	8.594697939	1.8662 E-04

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