

CONNECTION BETWEEN THE BMO- AND THE \mathcal{K}_Φ -SPACES

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1. Introduction

It is well-known that the dual of the Hardy space \mathcal{H}_1 is the BMO-space [4]. The dual space of the Hardy space \mathcal{H}_p , where $1 \leq p \leq 2$ has been investigated by Garsia, A. M. [5]. Mogyoródi, J. [7] and the author of the present note [2] have investigated the dual space of the Hardy space generated by Young functions Φ . This paper consists of two parts. In the first part we show that the dual of the space \mathcal{H}_Φ is the space \mathcal{H}_Ψ , where (Φ, Ψ) is a pair of conjugate Young functions, Φ has finite power and has the form $\Phi(x) = \Phi_1(x^2)$, where Φ_1 itself is also a Young function. In the second part we prove that the BMO-space can be approximated by the class of the so called \mathcal{K}_Φ -spaces. Theorem 3.2. is a generalization of some results of [1] and [7].

2. Basic notations and definitions

Let $\varphi(t)$ be a nondecreasing and left-continuous function defined on $[0, +\infty)$ such that $\varphi(0) = 0$ and $\lim_{t \rightarrow +\infty} \varphi(t) = +\infty$. For $x \geq 0$ define

$$\Phi(x) = \int_0^x \varphi(t) dt .$$

Then Φ is a convex, continuous and increasing function. Φ is called Young function. The power of the Young function Φ is defined by the formula

$$p = \sup_{x>0} [x\varphi(x)/\Phi(x)] .$$

The generalized inverse function ψ of the function φ is defined as follows

$$\psi(u) = \begin{cases} 0, & \text{if } u = 0 \\ \sup\{s > 0 : \varphi(s) < u\}, & \text{if } u > 0 \end{cases}$$

It is easy to see that ψ is also nondecreasing, left-continuous and $\lim_{u \rightarrow +\infty} \psi(u) = +\infty$. Then the Young function

$$\Psi(x) = \int_0^x \psi(u) du$$

is called the conjugate function of the Young function Φ . Its power is defined as that of Φ by the formula

$$q = \sup_{x>0} [x\psi(x)/\Psi(x)]$$

Let (Ω, \mathcal{F}, P) be a fixed probability space. Denote

$$L_0(\Omega, \mathcal{F}, P) = \{X : X : \Omega \rightarrow R; X \text{ if } \mathcal{F}\text{-measurable}\} .$$

DEFINITION 2.1. Let Φ be a Young function. The Orlicz space generated by Φ is defined as follows

$$L_\Phi = L_\Phi(\Omega, \mathcal{F}, P) = \{X \in L_0 : \exists a > 0 : E\Phi(a^{-1} | X |) \leq 1\}$$

If $X \in L_\Phi$ then the Luxemburg norm of X is defined by

$$\|X\|_\Phi = \inf\{a > 0 : E\Phi(a^{-1} | X |) \leq 1\} .$$

It is well-known that L_Φ equipped with the norm $\|\cdot\|_\Phi$ is a Banach-space. Further,

$$\|X\|_\Phi \leq \sup_{E\Psi(|Y|) \leq 1} E(XY) \leq 2\|X\|_\Phi .$$

where Ψ is the conjugate Young function of the function Φ (see [8]). The quantity $\sup_{E\Psi(|Y|) \leq 1} E(XY)$ is called the Orlicz-norm of X .

We refer to Krasnoselskii and Rutickii [6] for a complete treatment of the theory of Young functions.

Let $\mathcal{F}_0 \subset \mathcal{F}_1 \dots$ be an increasing sequence of subsigma-fields of F . We suppose that $\mathcal{F} = \sigma(\cup_{n=1}^\infty \mathcal{F}_n)$. Consider the random variable $X \in L_1(\Omega, \mathcal{F}, P)$ and the martingale $X_n = E(X | \mathcal{F}_n)$, $n \geq 0$, where for the sake of commodity we suppose that $X_0 = 0$ a.s. Denote $d_i = X_i - X_{i-1}$, $i \geq 1$, $d_0 = 0$.

DEFINITION 2.2. (see [7])

We say that the random variable $X \in L_1$ belongs to the Hardy-space \mathcal{H}_Φ generated by the Young function Φ if the quadratic variation $S = (\sum_{i=1}^\infty d_i^2)^{1/2}$ belongs to L_Φ . We define

$$\|X\|_{\mathcal{H}_\Phi} = \|S\|_\Phi$$

The space \mathcal{H}_Φ equipped with the norm $\|\cdot\|_{\mathcal{H}_\Phi}$ is a Banach space.

3. The dual of the Hardy space \mathcal{H}_Φ

In [2] we proved that the dual of \mathcal{H}_Φ is \mathcal{H}_Ψ in the case when both Φ and Ψ have finite power (in this case the authors of [1] have shown that the \mathcal{H}_Ψ -and the \mathcal{K}_Ψ -spaces coincide).

In this paper we shall prove that under some additional condition imposed on the function Φ , our result remains valid without the condition that Ψ , the conjugate of Φ , has finite power

We begin this section by proving the following

Lemma 3.1. *Let (Φ, Ψ) be a pair of conjugate Young functions. Suppose that Φ has the form $\Phi(x) = \Phi_1(x^2)$ where Φ_1 itself is a Young function and Φ_1 has finite power p . Let*

$$\begin{aligned} \delta\mathcal{M}_\Phi &= \{\Theta = (\Theta_n)_{n \geq 1} : \Theta_n \in L_\Phi(\Omega, \mathcal{F}_n, P); \|\Theta\|_{\delta\mathcal{M}_\Phi} = \\ &= \|(\sum_{n=1}^{\infty} \Theta_n^2)^{1/2}\|_\Phi < +\infty\} \end{aligned}$$

and let $\Lambda(\Theta)$ be a bounded and linear functional on $\delta\mathcal{M}_\Phi$, i.e.

$$|\Lambda(\Theta)| \leq B \cdot \|\Theta\|_{\delta\mathcal{M}_\Phi}, \quad \forall \Theta \in \delta\mathcal{M}_\Phi,$$

where $0 < B < +\infty$ is a finite constant. Then there exists $\sigma \in \delta\mathcal{M}_\Psi$ satisfying the conditions

$$(3.1.) \quad \begin{aligned} &\|(\sum_{n=1}^{\infty} \sigma_n^2)^{1/2}\|_\Psi \leq \sqrt{p}B \\ &\|[\sum_{n=1}^{\infty} E^2(\sigma_n | \mathcal{F}_{n-1})]^{1/2}\|_\Psi \leq (\sqrt{p} + 1)B, \end{aligned}$$

such that

$$\Lambda(\Theta) = \sum_{n=1}^{\infty} E(\Theta_n \cdot \sigma_n), \quad \forall \Theta \in \delta\mathcal{M}_\Phi.$$

Proof. First we remark that Φ has finite power if and only if so does Φ_1 . Now for any fixed integer $n \geq 1$ and any $X \in L_\Phi(\Omega, \mathcal{F}_n, P)$ let $\Theta_X = (\Theta_1^X, \Theta_2^X, \dots)$, where $\Theta_i^X = X$, if $i = n$ and $\Theta_i^X = 0$, if $i \neq n$. Obviously, $\Theta_X \in \delta\mathcal{M}_\Phi$ and $\|\Theta_X\|_{\delta\mathcal{M}_\Phi} = \|X\|_\Phi$. Then

$$|\Lambda(\Theta_X)| \leq B \|\Theta_X\|_{\delta\mathcal{M}_\Phi} = B \|X\|_\Phi$$

Since Φ has finite power, there exists $\sigma_n \in L_\Psi(\Omega, \mathcal{F}_n, P)$ such that

$$\Lambda(\Theta_X) = E(X \cdot \sigma_n), \quad \forall X \in L_\Phi(\Omega, \mathcal{F}_n, P).$$

Consider now an element $\Theta^{(n)} = (\Theta_1, \Theta_2, \dots)$, for which $\Theta_i = 0$ if $i > n$. By the linearity of the functional Λ we have

$$\Lambda(\Theta^{(n)}) = \sum_{i=1}^n E(\Theta_i \sigma_i)$$

where $\sigma_i \in L_\Psi(\Omega, \mathcal{F}_i, P)$, $i = 1, 2, \dots$. From the boundedness of Λ it follows that

$$|\Lambda(\Theta) - \Lambda(\Theta^{(n)})| \leq B \|(\sum_{i=n+1}^{\infty} \Theta_i^2)^{1/2}\|_\Phi.$$

The right-hand side of this inequality tends to zero as $n \rightarrow +\infty$. Consequently, we deduce

$$\Lambda(\Theta) = \lim_{n \rightarrow +\infty} \Lambda(\Theta^{(n)}) = \lim_{n \rightarrow +\infty} \sum_{i=1}^n E(\Theta_i \sigma_i) = \sum_{i=1}^{\infty} E(\Theta_i \sigma_i).$$

We prove that

$$\left(\sum_{i=1}^{\infty} \sigma_i^2\right)^{1/2} \in L_\Psi.$$

For this purpose we observe that the space $\delta\mathcal{M}_\Phi$ can be imbedded into the following larger space:

$$\begin{aligned} \widehat{\delta\mathcal{M}}_\Phi &= \{\Theta = (\Theta_n)_{n \geq 1} : \Theta_n \in L_\Phi(\Omega, \mathcal{F}, P), \|\Theta\|_{\widehat{\delta\mathcal{M}}_\Phi} = \\ &= \|(\sum_{i=1}^{\infty} \Theta_i^2)^{1/2}\|_\Phi < +\infty\} \end{aligned}$$

Then the functional Λ can be extended onto $\widehat{\delta\mathcal{M}}_\Phi$ as follows:

$$(3.2) \quad \Lambda(\Theta) = \sum_{n=1}^{\infty} E[E(\Theta_n | \mathcal{F}_n)\sigma_n], \quad \forall \Theta \in \widehat{\delta\mathcal{M}}_{\Phi}$$

This can be done since we can show that $(E(\Theta_n | \mathcal{F}_n))_{n \geq 1} \in \delta\mathcal{M}_{\Phi}$ if $(\Theta_n)_{n \geq 1} \in \widehat{\delta\mathcal{M}}_{\Phi}$.

Indeed, denote

$$a = \left\| \left(\sum_{n=1}^{\infty} \Theta_n^2 \right)^{1/2} \right\|_{\Phi}$$

We can suppose without any restriction that $a > 0$. Then

$$(3.3) \quad E \left[\Phi_1 \left(a^{-2} \cdot \sum_{n=1}^{\infty} \Theta_n^2 \right) \right] = E \left[\Phi \left(a^{-1} \left(\sum_{n=1}^{\infty} \Theta_n^2 \right)^{1/2} \right) \right] \leq 1.$$

So $\sum_{n=1}^{\infty} \Theta_n^2$ belongs to L_{Φ_1} and $\left\| \sum_{n=1}^{\infty} \Theta_n^2 \right\|_{\Phi_1} \leq a^2$. From the inequality $E^2(\Theta_n | \mathcal{F}_n) \leq E(\Theta_n^2 | \mathcal{F}_n)$ we deduce that

$$(3.4) \quad \begin{aligned} E \left(\Phi_1 \left[p^{-1} a^{-2} \sum_{n=1}^{\infty} E^2(\Theta_n | \mathcal{F}_n) \right] \right) &\leq \\ &\leq E \left(\Phi_1 \left[p^{-1} a^{-2} \sum_{n=1}^{\infty} E(\Theta_n^2 | \mathcal{F}_n) \right] \right) \end{aligned}$$

holds. On the other hand using the convexity inequality of Burkholder-Davis-Gundy (see [3]) we can write (remarking that the power of Φ_1 is finite)

$$(3.5) \quad E \left(\Phi_1 \left[p^{-1} a^{-2} \sum_{n=1}^{\infty} E(\Theta_n^2 | \mathcal{F}_n) \right] \right) \leq E \left[\Phi_1 \left(a^{-2} \sum_{n=1}^{\infty} \Theta_n^2 \right) \right].$$

Combining (3.3), (3.4) and (3.5) we have

$$E\left[\Phi_1\left(p^{-1}a^{-2}\sum_{n=1}^{\infty}[E(\Theta_n|\mathcal{F}_n)]^2\right)\right]\leq 1$$

In the language of Φ this can be written in the form

$$\begin{aligned} E\left[\Phi\left(p^{-1/2}a^{-1}\left[\sum_{n=1}^{\infty}E^2(\Theta_n|\mathcal{F}_n)\right]^{1/2}\right)\right] &= \\ &= E\left(\Phi_1\left[p^{-1}a^{-2}\sum_{n=1}^{\infty}E^2(\Theta_n|\mathcal{F}_n)\right]\right)\leq 1. \end{aligned}$$

This means that

$$\left[\sum_{n=1}^{\infty}E^2(\Theta_n|\mathcal{F}_n)\right]^{1/2}\in L_\Phi$$

and

$$\left\|\left[\sum_{n=1}^{\infty}E^2(\Theta_n|\mathcal{F}_n)\right]^{1/2}\right\|_\Phi\leq\sqrt{p}\cdot a=\sqrt{p}\left\|\left(\sum_{n=1}^{\infty}\Theta_n^2\right)^{1/2}\right\|_\Phi.$$

Then we can estimate (3.2) as follows

$$(3.6) \quad |\Lambda(\Theta)|\leq\sqrt{p}\cdot B\left\|\left(\sum_{n=1}^{\infty}\Theta_n^2\right)^{1/2}\right\|_\Phi,\quad\forall\Theta\in\widehat{\delta}\mathcal{H}_\Phi.$$

We use the Orlicz norm to estimate the L_Ψ -norm of $\left(\sum_{n=1}^{\infty}\sigma_n^2\right)^{1/2}$.

Denote $\gamma^2=\sum_{n=1}^{\infty}\sigma_n^2$. Then

$$\|\gamma\|_\Psi=\sup_{\substack{E\Phi(|X|)\leq 1 \\ X\in L_\Phi(\Omega,\mathcal{F},P)}}E(X\gamma)$$

Define

$$Y = \begin{cases} X\gamma^{-1}, & \text{if } \gamma \neq 0 \\ 0, & \text{if } \gamma = 0. \end{cases}$$

Consider the element $\Theta^{(n)} = (Y\sigma_1, \dots, Y\sigma_n, 0, 0, \dots)$. Obviously, $\Theta^{(n)} \in \widehat{\delta\mathcal{M}}$ and $\|\Theta^{(n)}\|_{\widehat{\delta\mathcal{M}_\bullet}} \leq 1$, since

$$E\left(\Phi\left[\left(\sum_{i=1}^n Y^2 \sigma_i^2\right)^{1/2}\right]\right) \leq E[\Phi(|X|)] \leq 1.$$

We can write

$$\begin{aligned} E(X\gamma) &= E(Y\gamma^2) = \sum_{i=1}^n E(Y\sigma_i^2) = \\ &= \sum_{i=1}^n E\left[E(Y\sigma_i \mid \mathcal{F}_i)\sigma_i\right] = \Lambda(\Theta^{(n)}). \end{aligned}$$

Using (3.6) we have

$$|E(X\gamma)| = |\Lambda(\Theta^{(n)})| \leq \sqrt{p} \cdot B \|\Theta^{(n)}\|_{\widehat{\delta\mathcal{M}_\bullet}} \leq \sqrt{p} \cdot B.$$

Consequently,

$$\|\gamma\|_{\Psi} \leq \sqrt{p} \cdot B$$

and, letting $n \rightarrow +\infty$, finally we have

$$\left\| \left(\sum_{n=1}^{\infty} \sigma_n^2 \right)^{1/2} \right\|_{\Psi} \leq \sqrt{p} \cdot B.$$

For arbitrary $\Theta = (\Theta_1, \Theta_2, \dots) \in \delta\mathcal{M}_\Phi$ define the element

$$\bar{\Theta} = (0, \Theta_1, \Theta_2, \dots) \in \delta\mathcal{M}_\Phi$$

and consider the functional $\bar{\Lambda}$ defined by the formula

$$\bar{\Lambda}(\Theta) = \Lambda(\bar{\Theta}), \quad \forall \Theta \in \delta\mathcal{M}_\Phi.$$

Then $\bar{\Lambda}$ is also a bounded and linear functional on $\delta\mathcal{X}_\Phi$ since

$$|\bar{\Lambda}(\Theta)| = |\Lambda(\bar{\Theta})| \leq B \|\bar{\Theta}\|_{\delta\mathcal{X}_\Phi} = B \|\Theta\|_{\delta\mathcal{X}_\Phi}.$$

As we have shown above, there exists $\mu = (\mu_n)_{n \geq 1} \in \delta\mathcal{X}_\Psi$ satisfying $\|(\sum_{n=1}^{\infty} \mu_n^2)^{1/2}\|_\Psi \leq \sqrt{p} \cdot B$ such that

$$(3.7) \quad \bar{\Lambda}(\Theta) = \sum_{n=1}^{\infty} E(\Theta_n \mu_n).$$

But

$$(3.8) \quad \begin{aligned} \bar{\Lambda}(\Theta) = \Lambda(\bar{\Theta}) &= \sum_{n=1}^{\infty} E(\Theta_n \sigma_{n+1}) = \\ &= \sum_{n=1}^{\infty} E\left[\Theta_n \cdot E(\sigma_{n+1} \mid \mathcal{F}_n)\right]. \end{aligned}$$

From (3.7) and (3.8) we deduce that

$$E(\sigma_{n+1} \mid \mathcal{F}_n) = \mu_n \quad \text{a.s.}$$

This implies

$$\begin{aligned} \left\| \left[\sum_{n=0}^{\infty} (\sigma_{n+1} \mid \mathcal{F}_n) \right]^{1/2} \right\|_\Psi &\leq \| |E(\sigma_1 \mid \mathcal{F}_0)| + \\ &\left[\sum_{n=1}^{\infty} E^2(\sigma_{n+1} \mid \mathcal{F}_n) \right]^{1/2} \|_\Psi \leq B + \sqrt{p} \cdot B = (\sqrt{p} + 1)B. \end{aligned}$$

Therefore, the proof is complete.

Now we present the main result which we formulate in

Theorem 3.2. *Let (Φ, Ψ) be a pair of conjugate Young functions.*

a/ For every $X \in \mathcal{X}_\Phi, Y \in \mathcal{X}_\Psi$ we have for arbitrary $n \geq 1$

$$|E(X_n Y_n)| \leq 2 \|X_n\|_{\mathcal{X}_\Phi} \|Y_n\|_{\mathcal{X}_\Psi}.$$

Further, $\lim_{n \rightarrow +\infty} E(X_n, Y_n)$ exists, it is finite and

$$|\lim_{n \rightarrow +\infty} E(X_n Y_n)| \leq 2 \|X\|_{\mathcal{X}_\Phi} \|Y\|_{\mathcal{X}_\Psi}.$$

b/ Suppose that Φ has the form $\Phi(x) = \Phi_1(x^2)$, where $\Phi_1(x)$ itself is a Young function having finite power p . If F is a bounded and linear functional on \mathcal{X}_Φ , i.e.

$$|F(x)| \leq B \|X\|_{\mathcal{X}_\Phi} \quad (B > 0 \text{ constant})$$

then there exists $Y \in \mathcal{X}_\Psi$ such that $\|Y\|_{\mathcal{X}_\Psi} \leq \sqrt{2}(2\sqrt{p} + 1)B$ and

$$\lim_{n \rightarrow +\infty} E(X_n Y_n) = F(X), \quad \forall x \in \mathcal{X}_\Phi.$$

Proof. a/ For any $X \in \mathcal{X}_\Phi, Y \in \mathcal{X}_\Psi$ denote $\Delta X_i = X_i - X_{i-1}, i = 1, 2, \dots; \Delta Y_j = Y_j - Y_{j-1}, j = 1, 2, \dots$. The Cauchy-Schwartz inequality gives

$$\left(\sum_{i=1}^n \Delta X_i \Delta Y_i \right)^2 \leq \left[\sum_{i=1}^n (\Delta X_i)^2 \right] \left[\sum_{j=1}^n (\Delta Y_j)^2 \right]$$

and from this by Hölder's inequality for conjugate Young functions

$$\begin{aligned} |E(X_n Y_n)| &= \left| \sum_{i=1}^n E(\Delta X_i \Delta Y_i) \right| = \left| E\left(\sum_{i=1}^n \Delta X_i \Delta Y_i \right) \right| \leq \\ &\leq E\left(\left[\sum_{i=1}^n (\Delta X_i)^2 \right]^{1/2} \left[\sum_{j=1}^n (\Delta Y_j)^2 \right]^{1/2} \right) \leq \\ &\leq 2 \left\| \left[\sum_{i=1}^n (\Delta X_i)^2 \right]^{1/2} \right\|_\Phi \left\| \left[\sum_{j=1}^n (\Delta Y_j)^2 \right]^{1/2} \right\|_\Psi. \end{aligned}$$

It is easy to see that $[E(X_n Y_n)]_{n \geq 1}$ is a Cauchy sequence, since for $m \geq n$ we have

$$\begin{aligned} |E(X_m Y_m) - E(X_n Y_n)| &= |E\left(\sum_{i=n+1}^m \Delta X_i \Delta Y_i\right)| \leq \\ &\leq 2 \left\| \left[\sum_{i=n+1}^m (\Delta X_i)^2 \right]^{1/2} \right\|_\Phi \left\| \left[\sum_{j=n+1}^m (\Delta Y_j)^2 \right]^{1/2} \right\|_\Psi \rightarrow 0 \end{aligned}$$

as $n \rightarrow +\infty$. Therefore $\lim_{n \rightarrow +\infty} E(X_n Y_n)$ exists and

$$\left| \lim_{n \rightarrow +\infty} E(X_n Y_n) \right| \leq 2 \|X\|_{\mathcal{K}_\Phi} \|Y\|_{\mathcal{K}_\Psi}.$$

b/ Suppose that F is a linear functional on \mathcal{K}_Φ such that

$$|F(X)| \leq B \|X\|_{\mathcal{K}_\Phi}, \quad \forall X \in \mathcal{K}_\Phi.$$

Then \mathcal{K}_Φ can be inbedded in the space $\delta\mathcal{K}_\Phi$ and the definition of F can be extended onto $\delta\mathcal{K}_\Phi$ with the same bound as that of F . By Lemma 3.1. there exists $\sigma = (\sigma_n)_{n \geq 1} \in \delta\mathcal{K}_\Psi$ such that

$$F(X) = \sum_{n=1}^{\infty} E(\Delta X_n \sigma_n).$$

Consider the martingale

$$Y_n = \sum_{i=1}^n \left[\sigma_i - E(\sigma_i | \mathcal{F}_{i-1}) \right].$$

Then $(Y_n)_{n \geq 1}$ is a Cauchy sequence in \mathcal{K}_Ψ , since

$$\begin{aligned} \|Y_m - Y_n\|_{\mathcal{K}_\Psi} &= \left\| \left[\sum_{i=n+1}^m (\Delta Y_i)^2 \right]^{1/2} \right\|_\Psi \leq \\ &\leq \sqrt{2} \left(\left\| \left[\sum_{i=n+1}^m \sigma_i^2 \right]^{1/2} \right\|_\Psi + \left\| \left[\sum_{i=n+1}^m E^2(\sigma_i | \mathcal{F}_{i-1}) \right]^{1/2} \right\|_\Psi \right) \rightarrow 0 \end{aligned}$$

So, Y_n must converge to

$$Y = \sum_{i=1}^{\infty} [\sigma_i - E(\sigma_i \mid \mathcal{F}_{i-1})]$$

belonging to \mathcal{H}_{Ψ} . Finally,

$$\begin{aligned} \|Y\|_{\mathcal{H}_{\Psi}} &\leq \sqrt{2} \left(\left\| \left(\sum_{i=1}^{\infty} \sigma_i^2 \right)^{1/2} \right\|_{\Psi} + \left\| \left[\sum_{i=1}^{\infty} E^2(\sigma_i \mid \mathcal{F}_{i-1}) \right]^{1/2} \right\|_{\Psi} \right) \leq \\ &\leq \sqrt{2} [\sqrt{p}B + (\sqrt{p} + 1)B] = \sqrt{2}(2\sqrt{p} + 1)B. \end{aligned}$$

This completes the proof.

The BMO-space and the \mathcal{K}_{Φ} -spaces

In this section we present the connection between the BMO-space and the \mathcal{K}_{Φ} -spaces. We show that the BMO-space can be approximated in some sense by the class of the \mathcal{K}_{Φ} -spaces. We recall some definitions.

DEFINITION 4.1.

a/ Let $X \in L_1$. Consider the martingale $X_n = E(X \mid \mathcal{F}_n)$, $n \geq 0$, $x_0 = 0$ a.s. We say that X belongs to BMO iff

$$\sup_{n \geq 1} \|E(|X - X_{n-1}| \mid \mathcal{F}_n)\|_{\infty} < +\infty.$$

b/ Let Φ be a Young function and let $X \in L_1$. Consider the set

$$\Gamma_X^{\Phi} = \{\gamma : \gamma \in L_{\Phi}, E(|X - X_{n-1}| \mid \mathcal{F}_n) \leq E(\gamma \mid \mathcal{F}_n) \text{ a.s., } \forall n \geq 1\}.$$

We say that $X \in \mathcal{K}_{\Phi}$ iff Γ_X^{Φ} is non empty and in this case we set

$$\|X\|_{\mathcal{K}_{\Phi}} = \inf_{\gamma \in \Gamma_X^{\Phi}} \|\gamma\|_{\Phi}$$

The following theorem shows that the K_Φ -space is the direct generalization of BMO.

Theorem 4.1. *Let (Φ, Ψ) be a pair of conjugate Young functions. Suppose that Ψ has finite power q . Then $X \in K_\Phi$ if and only if $\beta^* = \sup_{n \geq 1} E(|X - X_{n-1}| | \mathcal{F}_n) \in L_\Phi$ and*

$$\|X\|_{K_\Phi} \leq \|\beta^*\|_\Phi \leq q\|X\|_{K_\Phi}.$$

Proof. If $\beta^* \in L_\Phi$ then $E(|X - X_{n-1}| | \mathcal{F}_n) \leq E(\beta^* | \mathcal{F}_n)$ a.s. Consequently, $X \in K_\Phi$ and $\|X\|_{K_\Phi} \leq \|\beta^*\|_\Phi$.

Conversely, suppose that $X \in K_\Phi$. Then for all $n \geq 1$ we have

$$E(|X - X_{n-1}| | \mathcal{F}_n) \leq E(\gamma | \mathcal{F}_n) \text{ a.s.,}$$

where $\gamma \in \Gamma_X^\Phi$ is arbitrary. Let $\gamma^* = \sup_{n \geq 1} E(\gamma | \mathcal{F}_n)$. Then by the maximal lemma (see [9]) $\gamma^* \in L_\Phi$ and

$$\|\gamma^*\|_\Phi \leq q \sup_{n \geq 1} \|E(\gamma | \mathcal{F}_n)\|_\Phi \leq q\|\gamma\|_\Phi.$$

On the other hand

$$\beta^* = \sup_{n \geq 1} E(|X - X_{n-1}| | \mathcal{F}_n) \leq \gamma^*$$

This implies that $\beta^* \in L_\Phi$ and

$$\|\beta^*\|_\Phi \leq \|\gamma^*\|_\Phi \leq q\|\gamma\|_\Phi$$

Since the above inequality holds for every γ belonging to Γ_X^Φ we deduce that

$$\|\beta^*\|_\Phi \leq q\|X\|_{K_\Phi}.$$

This completes the proof.

Theorem 4.2. *Let \mathcal{Y} denote the class of all Young functions Ψ whose conjugate Young function Φ has finite power. Then*

$$\text{BMO} = \bigcap_{\Psi \in \mathcal{Y}} K_{\Psi} .$$

Proof. Obviously, $\text{BMO} \subset \bigcap_{\Psi \in \mathcal{Y}} K_{\Psi}$. Now we suppose that $Y \notin \text{BMO}$. This means that $\beta^* = \sup_{n \geq 1} E(|Y - Y_{n-1}| | \mathcal{F}_n) \notin L_{\infty}$. We can suppose that $P(\beta^* < +\infty) = 1$, since in the contrary case $\beta^* \notin L_{\Psi}$ whatever be the Young function Ψ . Denote $a_n = P(\beta^* \geq n)$. Then $a_n > 0$ and $a_n \downarrow 0$ as $n \rightarrow +\infty$. We define the function

$$\Psi(x) = \int_0^x \psi(t) dt$$

as follows: let $\psi(0) = 0$, $\psi(1) = 2/a_2, \dots, \psi(n) = 2/a_{(n+1)^2}, \dots$. Further, let $\psi(t)$ be linear in every interval $[n, n+1]$. Consequently, $\Psi(x)$ is a Young function. We shall show that $\beta^* \notin L_{\Psi}$. For each positive integer $k \geq 1$ we have

$$\begin{aligned} E[\Psi(k^{-1}\beta^*)] &\geq E[\Psi(k^{-1}\beta^*)\chi(\beta^* \geq k^2)] \geq \\ &\geq \Psi(k)P(\beta^* \geq k^2) = \Psi(k)a_k, \end{aligned}$$

By the definition of Ψ we can write

$$\Psi(k) = \int_0^k \psi(t) dt \geq \int_{k-1}^k \psi(t) dt \geq \psi(k-1) = 2/a_{k^2} .$$

Therefore,

$$E[\Psi(k^{-1}\beta^*)] \geq \Psi(k)a_k \geq (2/a_{k^2})a_k = 2.$$

This means that $\beta^* \notin L_{\Psi}$ and by Theorem 4.1. $Y \notin K_{\Psi}$ provided that Φ has finite power.

Now, if the conjugate Young function Φ of the function Ψ has infinite power, then remarking that $\sup_{0 < x \leq 1} [\varphi(2x)/\varphi(x)] = a < +\infty$ we can define another function φ_1 as follows:

$$\varphi_1(x) = \begin{cases} \varphi(x) & \text{if } 0 \leq x \leq 1. \\ \varphi(x) & \text{if } \varphi(x) \leq a\varphi(x/2) \text{ and } 2^i < x \leq 2^{i+1}, \\ \varphi(x/2) & \text{if } \varphi(x) > a\varphi(x/2) \text{ and } 2^i < x \leq 2^{i+1}, \end{cases}$$

$i = 0, 1, \dots$ and

$$\Phi_1(x) = \int_0^x \varphi_1(t) dt .$$

Obviously, Φ_1 has finite power and $\Phi_1(x) \leq \Phi(x)$, since $\varphi_1(x) \leq \varphi(x)$. This also implies that $\Psi_1(x) \geq \Psi(x)$, where $\Psi_1(x)$ is the conjugate function of the function $\Phi_1(x)$. Consequently, $\beta^* \notin L_{\Psi_1}$, which means that $Y \notin \bigcap_{\Psi \in \mathcal{Y}} K_\Psi$

This proves the assertion.

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