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ON THE CONVERGENCE OF CERTAIN ITERATIONS TO THE FIXED POINTS OF NONLINEAR OPERATORS

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Abstract. Our results answer to the following question: given that a mapping has a fixed point, when is it true that the Newton iterates produce a sequence of nearby points which converge to the fixed point? We assume only that the nonlinear operator has a Hölder continuous Fréchet-derivative at the fixed point.

1. Introduction.

This paper is devoted to a generalization of a theorem of Jon Rockne [5] concerning fixed points. Consider an equation

$$F(x) = 0 \tag{1}$$

where F is a nonlinear operator between two Banach spaces X_1 and X_2. Under certain conditions, Newton's method

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n), \quad n = 0, 1, 2, \dots \tag{2}$$

produces a sequence which converges quadratically to a solution x^* of (1). One of the assumptions of this method is that F is twice Fréchet-differentiable in some ball around the initial iterate. However there are many interesting problems already in the literature where the operator is only once Fréchet-differentiable [1], [5].

Jon Rockne [5] has given some conditions under which when F has Hölder continuous Fréchet derivatives then iteration (2) converges to a solution x^* of (1).

The theorems proved here are concerned with the following question: given an operator has a fixed point x^*, when is it true that iteration (2) produces a sequence of nearby points which converge to x^*? Such a question is clearly of interest in numerical analysis since many numerical problems can be reduced to the problem of locating fixed points.

Finally, we provide two examples, one from the real scalar case and the other using a second order differential equation.

2. Preliminaries.

We assume that F is once Fréchet differentiable [2], [4] and $F'(x)$ is the first Fréchet-derivative at a point x. It is well known that $F'(x) \in L(X_1, X_2)$, the space of bounded linear operators from X_1 to X_2. We say that the Fréchet derivative $F'(x)$ is Hölder continuous over a domain D if for some $c > 0$, $p \in [0,1]$, and all $x, y \in D$,

$$\|F'(x) - F'(y)\| \leq c\|x - y\|^p. \tag{3}$$

In this case we say that $F'(\bullet) \in H_D(c,p)$.

We now include the following known result because we feel that the literature is not too easily accessible.

Lemma. *Let $F : X_1 \to X_2$ and $D \subset X_1$. Assume D is open and that $F'(\bullet)$ exists at each point of D. If for some convex $D_0 \subseteq D$ we have $F'(\bullet) \in H_{D_0}(c,p)$, then for all $x, y \in D_0$*

$$\|F(x) - F(y) - F'(x)(x - y)\| \leq \frac{c}{1+p}\|x - y\|^{1+p}. \tag{4}$$

Proof. Using standard properties of the integral and the fact that D_0 is convex we have:

$$F(x)-F(y)-F'(y)(x-y) = \int_0^1 (F'(y+t(x-y))-F'(y))(x-y)dt \, .$$

By taking norms we obtain

$$\|F(x) - F(y) - F'(y)(x - y)\| \leq \int_0^1 \|F'(y + t(x - y))-$$

$$- F'(y)\| \cdot \|x - y\| \cdot dt$$

$$\leq \int_0^1 c\|y + t(x - y) - y\|^p \cdot \|x - y\| \cdot dt$$

$$= \frac{c}{1+p}\|x - y\|^{1+p} \, ,$$

and that completes the proof of the lemma.

3. Main results.

Let x^* be a simple solution of (1) in the sense that $F'(x^*)$ has a bounded inverse. Then there exists $\epsilon > 0$ such that $F'(x)$ has a bounded inverse for all $x \in U(x^*, \epsilon) = \{x \in X_1 \mid \|x - x^*\| < \epsilon\}$. Set $b(x) = \|F'(x)^{-1}\|$ for $x \in U(x^*, \epsilon)$.

We can now prove the following:

Theorem 1. *Let $F : X_1 \to X_2$ and $D \subset X_1$.*

Assume

(a) *$x^* \in D$ is a simple solution of (1);*

(b) *there exists* $\epsilon, b > 0$ *such that*

$$b(x) \leq b \text{ for all } x \in U(x^*, \epsilon).$$

(c) *There exists some convex set* D_0 *with* $x^* \in D_0 \subset D$ *and some* $\epsilon_1 > 0$, *with* $0 < \epsilon_1 < \epsilon$ *such that* $F'(\bullet) \in H_{D_0}(c,p)$ *for all* $x, y \in D_0$ *and* $U(x^*, \epsilon_1) \subset D_0$.

(d) *The following estimate holds:*

$$0 < k < 1 \ ,$$

where

$$k = \frac{cb}{(p+1)^2} \ .$$

Fix $\epsilon_2 > 0$ such that

$$0 < \epsilon_2 < \min\left[\epsilon_1, \left[\frac{1}{k}\right]^{\frac{1}{p}}\right] \text{ if } p \neq 0 \text{ and}$$

$$0 < \epsilon_2 < \epsilon_1 \text{ if } p = 0 \ .$$

Then if $x_0 \in \bar{U}(x^*, \epsilon_2)$, the iterates x_n, $n = 0, 1, 2, \ldots$ given by (2) are well defined, remain in $\bar{U}(x^*, \epsilon_2)$ and converge to the unique solution x^* of (1) in $\bar{U}(x^*, \epsilon_2)$ with order of convergence $1 + p$.

Proof. Define the operator P on $\bar{U}(x^*, \epsilon_2)$ by

$$P(x) = x - F'(x)^{-1} F(x) \ .$$

The operator P is well defined on $\bar{U}(x^*, \epsilon_2)$ since $F'(x)^{-1}$ exists on $\bar{U}(x^*, \epsilon_2)$. Let $x \in \bar{U}(x^*, \epsilon_2)$, then

$$\begin{aligned} \|P(x) - x^*\| &= \|x - x^* - F'(x)^{-1} F(x)\| \\ &= \|F'(x)^{-1}[F'(x)(x - x^*) - (F(x) - F(x^*))]\| \end{aligned}$$

$$\leq \|F'(x)^{-1}\| \, \| \int_0^1 (F'(x) - F'(x + t(x^* - x)))(x - x^*)dt \|$$

$$< \frac{bc}{p+1} \|x - x^*\|^{p+1} \int_0^1 t^p \, dt$$

$$\leq k\|x - x^*\|^{p+1} < \epsilon_2 \tag{5}$$

by the choice of ϵ_2. Therefore P maps $\bar{U}(x^*, \epsilon_2)$ into itself and if $x_0 \in \bar{U}(x^*, \epsilon_2)$, all the iterates x_n, $n = 0, 1, 2, \ldots$ given by (2) remain in $\bar{U}(x^*, \epsilon_2)$.

Moreover,

$$x_{n+1} - x^* = x_n - x^* - F'(x_n)^{-1} F(x_n) \; .$$

By taking norms in the above equation, as in (5) we obtain

$$\|x_{n+1} - x^*\| \leq k\|x_n - x^*\|^{p+1} \tag{6}$$

$$\leq k(k^{p+1}\|x_{n-1} - x^*\|^{p+1})$$

$$\ldots$$

$$\leq k^{n(1+p)+1}\|x_0 - x^*\| \tag{7}$$

Since $0 < k < 1$ it follows that $k^{n(p+1)+1} \to 0$ as $n \to \infty$. Therefore by (7) the sequence $\{x_n\}$, $n = 0, 1, 2, \ldots$ converges to x^* with order of convergence $1 + p$ and that completes the proof of the theorem.

Note that if $p = 1$ and the second Fréchet-derivative of F at x^* is bounded our approach can be used to prove a result similar to the one stated in theorem 1 for the usual Newton's method. The order of convergence will then be 2.

Leaving that to the motivated reader we show instead how we can increase the order of convergence of (2) to a simple solution x^* of (1).

For x_n, $n = 0,1,2,\ldots$ prechosen we introduce the iteration

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n) \\ x_{n+1} &= y_n - F'(x_n)^{-1}F(y_n),\ n = 0,1,2,\ldots \end{aligned} \tag{8}$$

We can prove the following:

Theorem 2. Assume:

(i) *the hypotheses (a), (b) and (c) of theorem 1 for F hold;*

(ii) *the following is true:*

$$0 < k_1 < 1$$

where

$$k_1 = \frac{(cb)^2}{(p+1)^3}\left[1 + \frac{cb}{(p+1)^2}\epsilon_1^p\right]^p .$$

Fix $\epsilon_2 > 0$ such that

$$0 < \epsilon_2 < \min\left[\epsilon_1, \left[\frac{1}{k_1}\right]^{\frac{1}{p}}\right] \text{ if } p \neq 0 \text{ and}$$

$$0 < \epsilon_2 < \epsilon_1 \text{ if } p = 0 .$$

Then if $x_0 \in \bar{U}(x^*, \epsilon_2)$, the iterates x_n, $n = 0,1,2,\ldots$ given by (8) are well defined, remain in $\bar{U}(x^*, \epsilon_2)$ and converge to the unique solution x^* of (1) in $\bar{U}(x^*, \epsilon_2)$ with order of convergence $1 + 2p$.

Proof. Exactly as in theorem 1 we show that the iterates given by (8) remain in $\bar{U}(x^*, \epsilon_2)$ if $x_0 \in \bar{U}(x^*, \epsilon_2)$.

We also have

$$\|x_{n+1} - x^*\| = \|y_n - x^* - F'(x_n)^{-1}F(x_n)\|$$

$$\leq \|F'(x_n)^{-1}\| \, \|\int_0^1 (F'(x_n) - F'(x^* + t(y_n - x^*)))dt\| \, \|y_n - x^*\|$$

$$\leq \frac{cb}{p+1}\|(x_n - x^*) + t(y_n - x^*)\|^p \|y_n - x^*\|$$

$$\leq \frac{cb}{p+1}(\|x_n - x^*\| + \|y_n - x^*\|)^p \|y_n - x^*\| \ . \tag{9}$$

Moreover,

$$\|y_n - x^*\| \leq \|F'(x_n)^{-1}\| \cdot$$

$$\cdot \left\| \int_0^1 (F'(x_n) - F'(x^* + t(x_n - x^*)))dt \right\| \|x_n - x^*\|$$

$$\leq \frac{cb}{p+1}\left\| \int_0^1 (1-t)^p dt \right\| \|x_n - x^*\|^{p+1}$$

$$\leq \frac{cd}{(p+1)^2}\|x_n - x^*\|^{p+1} \ . \tag{10}$$

Finally, by (9) and (10) we get

$$\|x_{n+1} - x^*\| \leq \frac{cb}{p+1}\left[\|x_n - x^*\| + \frac{cb}{(p+1)^2}\|x_n - x^*\|^{p+1}\right]^p \cdot$$

$$\cdot \frac{cb}{(p+1)^2}\|x_n - x^*\|^{p+1}$$

$$\leq \frac{(cb)^2}{(p+1)^3}\left[1 + \frac{cb}{(p+1)^2}\|x_n - x^*\|^p\right]^p \|x_n - x^*\|^{2p+1}$$

$$\leq \frac{(cb)^2}{(p+1)^3}\left[1 + \frac{cb}{(p+1)^2}\epsilon_1^p\right]^p \|x_n - x^*\|^{2p+1}$$

$$\leq k_1 \|x_n - x^*\|^{2p+1} \tag{11}$$

$$\leq k_1 \left[k_1^{2p+1}\|x_{n-1} - x^*\|^{2p+1}\right]$$

$$\cdots$$

$$\leq k_1^{n(1+2p)+1}\|x_0 - x^*\| \ . \tag{12}$$

Newton's method cannot be applied to the equation

$$F(x) = 0 .$$

We may not be able to evaluate the second Fréchet-derivative since it would involve the evaluation of quantities of the form x_i^{-p} and they may not exist.

Let $x \in \mathbf{R}^{n-1}$, $H \in \mathbf{R}^{n-1} \times \mathbf{R}^{n-1}$ and define the norms of x and H by

$$\|x\| = \max_{1 \leq j \leq n-1} | x_j |$$

$$\|H\| = \max_{1 \leq j \leq n-1} \sum_{k=1}^{n-1} | h_{jk} | .$$

For all $x, z \in \mathbf{R}^{n-1}$ for which $| x_i | > 0$, $| z_i | > 0$, $i = 1, 2, \ldots, n-1$ we obtain

$$\|F'(x) - F'(z)\| = \|\text{diag}\{(1+p)h^2(x_j^p - z_j^p)\}\|$$

$$= (1+p)h^2 \max_{1 \leq j \leq n-1} | x_j^p - z_j^p | \leq (1+p)h^2 [\max | x_j - z_j |]^p$$

$$= (1+p)h^2 \|x - z\|^p .$$

Therefore, under the assumptions of theorem 2, iteration (8) will converge to the solution x^* of (16), provided that such a solution x^* is known.

References

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