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SPLINE APPROXIMATIONS FOR SYSTEM OF SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. A method for obtaining spline function approximations for the solution of the system of nonlinear ordinary differential equations $y'' = f_1(x,y,z), z'' = f_2(x,y,z)$ with $y(x_0) = y_0, z(x_0) = z_0, y'(x_0) = y'_0, z'(x_0) = z'_0$ is presented. The spline functions approximating the solutions are polynomial splines. It is a one-step method $O(h^\alpha)$ in $y^{(i)}(x)$ and $z^{(i)}(x)$ where $i = 0,1$ and 2 and $0 < \alpha \leq 1$ assuming $f_1, f_2 \in C$.

Description of the method

Consider the following system of nonlinear ordinary differential equations

$$(1) \qquad y'' = f_1(x,y,z),\ y(x_0) = y_0,\ y'(x_0) = y'_0,$$

$$(2) \qquad z'' = f_2(x,y,z),\ z(x_0) = z_0,\ z'(x_0) = z'_0,$$

where $f_1, f_2 \in C([0,1] \times R^2)$.

Let $\triangle$ be the partition

$$\triangle : 0 = x_0 < x_1 \ldots < x_k < x_{k+1} < \ldots < x_n = 1$$

where $x_{k+1} - x_k = h < 1$ and $k = o(1)n - 1$.

Let L_1 and L_2 be the Lipschitz constants satisfied by the functions f_1 and f_2 respectively, i.e.,

$$|f_1(x, y_1, z_1) - f_1(x, y_2, z_2)| \leq L_1\{|y_1 - y_2| + |z_1 - z_2|\}, \tag{3}$$

$$|f_2(x, y_1, z_1) - f_2(x, y_2, z_2)| \leq L_2\{|y_1 - y_2| + |z_1 - z_2|\} \tag{4}$$

for all (x, y_1, z_1) and (x, y_2, z_2) in the domain of definition of f_1 and f_2.

Then, we define the spline functions approximating the solution $y(x)$ and $z(x)$ by $S_\triangle(x)$ and $\bar{S}_\triangle(x)$ where

$$S_\triangle(x) = S_k(x), \quad x_k \leq x \leq x_{k+1}, \quad k = o(1)n - 1 \tag{5}$$

and

$$\bar{S}_\triangle(x) = \bar{S}_k(x), \quad x_k \leq x \leq x_{k+1}, \quad k = o(1)n - 1. \tag{6}$$

Both $S_\triangle(x)$ and $\bar{S}_\triangle(x)$ are given from the following:

$$\begin{aligned} S_k(x) =& S_{k-1}(x_k) + S'_{k-1}(x_k)(x - x_k) + \\ &+ \frac{1}{2} f_1[x_k, S_{k-1}(x_k), \bar{S}_{k-1}(x_k)](x - x_k)^2 \end{aligned} \tag{7}$$

and

$$\begin{aligned} \bar{S}_k(x) =& \bar{S}_{k-1}(x_k) + \bar{S}'_{k-1}(x_k)(x - x_k) + \\ &+ \frac{1}{2} f_2[x_k, S_{k-1}(x_k), \bar{S}_{k-1}(x_k)](x - x_k)^2 \end{aligned} \tag{8}$$

where $k = o(1)n - 1, S_{-1}(x_0) = y_0$ and $\bar{S}_{-1}(x_0) = z_0$.

By construction, it is clear that $S_\triangle(x) \in C^1[0, 1]$.

Convergence

We are going to discuss the convergence of these spline approximants.

For all $x \in [x_k, x_{k+1}], k = o(1)n-1$ the exact solutions of (1) and (2) can be written - by means of Taylor's expansion - in the forms:

$$y(x) = y_k + y'_k(x - x_k) + \frac{1}{2}y''(\xi_k)(x - x_k)^2 \tag{9}$$

and

$$z(x) = z_k + z'_k(x - x_k) + \frac{1}{2}z''(\eta_k)(x - x_k)^2 \tag{10}$$

where $\xi_k, \eta_k \in (x_k, x_{k+1})$ and $k = o(1)n - 1$.

First, we estimate $|y(x) - S_0(x)|$ where $x_0 \leq x \leq x_1$.

Using both (7) and (9) for $k = 0$, we get

$$\begin{aligned} |y(x) - S_0(x)| &= \frac{1}{2}|y''(\eta_0) - y''_0||x - x_0|^2 \\ &\leq \frac{1}{2}h^2\omega(y'', h) = O(h^{2+\alpha}) \end{aligned} \tag{11}$$

where $\omega(y'', h)$ is the modulus of continuity of the function y''.

Also,we estimate $|y'(x) - S'_0(x)|$ and $|y''(x) - S''_0(x)|$ where $x_0 \leq x \leq x_1$.

Using both (7) and (9) for $k = 0$, we get:

$$\begin{aligned} |y'(x) - S'_0(x)| &= |y''(\eta_0) - y''_0||x - x_0| \\ &\leq h\omega(y'', h) = O(h^{1+\alpha}) \end{aligned} \tag{12}$$

and

$$\begin{aligned} |y''(x) - S''_0(x)| &= |y''(x) - y''_0| \\ &\leq \omega(y'', h) = O(h^{\alpha}) \end{aligned} \tag{13}$$

In a similar manner, using both (8) and (10) for $k = 0$ and the Lipschitz condition (4), it can be shown, for $x_0 \leq x \leq x_1$, that:

$$|z(x) - \bar{S}_0(x)| \leq \frac{1}{2}h^2\omega(z'', h) = O(h^{2+\alpha}), \tag{14}$$

$$|z'(x) - \bar{S}_0'(x)| \leq h\omega(z'', h) = O(h^{1+\alpha}) \tag{15}$$

and

$$|z''(x) - \bar{S}_0''(x)| \leq \omega(z'', h) = O(h^{\alpha}) \tag{16}$$

where $\omega(z'', h)$ is the modulus of continuity of the function z''.

In what follows we deal with the general subinterval $I_k = [x_k, x_{k+1}]$, $k = 1(1)n - 1$.

We estimate $|y(x) - S_k(x)|$.

Using (7), (9) and the Lipschitz condition (3), we get:

(17)

$$|y(x) - S_k(x)| \leq |y_k - S_{k-1}(x_k)| + |y'_k - S'_{k-1}(x_k)||x - x_k| + \\ + \frac{1}{2}|y''(\xi_k) - f_1[x_k, S_{k-1}(x_k), \bar{S}_{k-1}(x_k)]||x - x_k|^2$$

Now let,

$$U_1 = y''(\xi_k) - f_1[x_k, S_{k-1}(x_k), \bar{S}_{k-1}(x_k)].$$

Then, using the Lipschitz condition (3), we get

$$\begin{aligned} U_1 &\leq |y''(\xi_k) - y_k''| + |f_1(x_k, y_k, z_k) - \\ &\quad - f_1[x_k, S_{k-1}(x_k), \bar{S}_{k-1}(x_k)]| \\ &\leq \omega(y'', h) + L_1\{|y_k - S_{k-1}(x_k)| + \\ &\quad + |z_k - \bar{S}_{k-1}(x_k)|\} \end{aligned} \tag{18}$$

Using the fact that $S_\triangle(x), \bar{S}_\triangle(x) \in C^1[0,1]$ and the notations:

$$
\begin{aligned}
e(x) &= |y(x) - S_k(x)|, \\
e_k &= |y_k - S_k(x_k)|, \\
\bar{e}_k(x) &= |z(x) - \bar{S}_k(x)|, \\
\bar{e}_k &= |z_k - \bar{S}_k(x_k)|
\end{aligned}
\tag{19}
$$

then (18) becomes:

$$
U_1 \le \omega(y'', h) + L_1(e_k + \bar{e}_k) \tag{20}
$$

Using (17)-(20), we can easily get:

$$
e(x) \le (1 + c_0 h^2)e_k + c_0 h^2 \bar{e}_k + he'_k + \frac{1}{2}h^2\omega(h) \tag{21}
$$

where $\omega(h) = \max\{\omega(y'', h), \omega(z'', h)\}$ and $c_0 = \frac{1}{2}L_1$, is a constant independent of h.

Similarly, using (8), (10) and the Lipschitz condition (4), we can show that:

$$
\bar{e}(x) \le c_1 h^2 e_k + (1 + c_1 h^2)\bar{e}_k + h\bar{e}'_k + \frac{1}{2}h^2\omega(h) \tag{22}
$$

where $c_1 = \frac{1}{2}L_2$, is a constant independent of h.

We now estimate $|y'(x) - S'_k(x)|$. Thus using (7) and (9), we get:

$$
\begin{aligned}
|y'(x) - S'_k(x)| \le |y'_k - S'_{k-1}(x_k)| + |y''(\xi_k) - \\
- f_1[x_k, S_{k-1}(x_k), \bar{S}_{k-1}(x_k)]| \cdot |x - x_k|
\end{aligned}
\tag{23}
$$

Using (18) and (19), inequality (23) yields:

$$
e'(x) \le L_1 h e_k + L_1 h \bar{e}_k + e'_k + h\omega(h) \tag{24}
$$

Similarly, using (8) and (10), we can easily see that:

$$\bar{e}'(x) \le L_2 h e_k + L_2 h \bar{e}_k + \bar{e}'_k + h\omega(h) \tag{25}$$

To complete the convergence proof, we use the matrix inequality which is given in the following definition:

DEFINITION 1. Let $A = [a_{ij}], B = [b_{ij}]$ be two matrices of the same order, then we say that $A \le B$ iff

(i) both a_{ij} and b_{ij} are nonnegative,

(ii) $a_{ij} \le b_{ij} \quad \forall i, j$.

According to this definition and if we use the matrix notations:

$$E(x) = (e(x)\ \bar{e}(x)\ e'(x)\ \bar{e}'(x))^T$$
$$E_k = (e_k\ \bar{e}_k\ e'_k\ \bar{e}'_k)^T$$

we can write the estimations (21), (22), (24) and (25) in the form:

$$E(x) \le (I + hA)E_k + h\omega(h)B \tag{26}$$

$$\text{where } \begin{pmatrix} c_0 & c_0 & 1 & 0 \\ c_1 & c_1 & 0 & 1 \\ L_1 & L_1 & 0 & 0 \\ L_2 & L_2 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \\ 1 \end{pmatrix}$$

and I is the identity matrix of order 4.

Now, we give the following definition of the matrix norm.

DEFINITION 2. Let $T = [t_{ij}]$ be a mxn matrix, then we define $\| T \| = \max\limits_{i} \sum\limits_{j=1}^{m} |t_{ij}|$

Using this definition, we get

$$\| E(x) \| = \max(e(x)\ \bar{e}(x)\ e'(x)\ \bar{e}'(x)) \tag{27}$$

Since (26) is valid for all $x \in [x_k, x_{k+1}]$, $k = o(1)n - 1$, then the following inequalities hold true:

$$\| E(x) \| \leq (1 + h \| A \|) \| E_k \| + h\omega(h) \| B \|$$
$$(1 + h \| A \|) \| E_k \| \leq (1 + h \| A \|)^2 \| E_{k-1} \| + \\ + h\omega(h) \| B \| (1 + h \| A \|)$$
$$(1 + h \| A \|)^2 \| E_{k-1} \| \leq (1 + h \| A \|)^3 \| E_{k-1} \| + \\ + h\omega(h) \| B \| (1 + h \| A \|)^2$$
$$\dots \quad \dots$$
$$(1 + h \| A \|)^k \| E_1 \| \leq (1 + h \| A \|)^{k+1} \| E_0 \| + \\ + h\omega(h) \| B \| (1 + h \| A \|)^k$$

Adding R.H.S. and L.H.S. of these inequalities and noting that $\| E_0 \| = 0$, we get:

$$\| E(x) \| \leq c_2 \omega(h) \tag{28}$$

where $c_2 = \frac{\|B\|}{\|A\|}(e^{\|A\|} - 1)$, is a constant independent of h.

By definition (27), we get

$$e^{(i)}(x) \leq c_2 \omega(h) = O(h^\alpha), \quad i = 0, 1 \tag{29}$$

and

$$\bar{e}^{(i)}(x) \leq c_2 \omega(h) = O(h^\alpha), \quad i = 0, 1. \tag{30}$$

We now estimate $|y''(x) - S_k''(x)|$.

Using (7) and the Lipschitz condition (3), we get:

$$\begin{aligned} |y''(x) - S_k''(x)| = |y''(x) - f_1[x_k, S_{k-1}(x_k), \bar{S}_{k-1}(x_k)]| \\ \omega(y'', h) + L_1(e_k + \bar{e}_k) \end{aligned} \tag{31}$$

Now, using (29) and (30), we get:

$$e''(x) \leq c_3 \omega(h) = O(h^\alpha) \tag{32}$$

where $c_3 = 1 + 2L_1 c_2$, is a constant independent of h.

In a similar manner, using (4), (8), (29) and (30), we can easily get:

$$\bar{e}''(x) \equiv |z''(x) - \bar{S}_k''(x)| \leq c_4 \omega(h) = O(h^\alpha)$$

where $c_4 = 1 + 2L_2 c_2$, is a constant independent of h.

Thus, we have proved the following theorem:

Theorem. *Let $S_\Delta(x)$ and $\bar{S}_\Delta(x)$ be the approximate solutions to problem (1)-(2) given by equations (7)-(8), and let $f_1, f_2 \in C([x_0, x_n] \times R^2)$.*

Then, for all $x \in [x_0, x_1]$ we have:

$$|y^{(i)}(x) - S_0^{(i)}(x)| \leq ch^{2-i}\omega(h), \quad i = 0, 1, 2$$

and

$$|z^{(i)}(x) - \bar{S}_0^{(i)}(x)| \leq Kh^{2-i}\omega(h), \quad i = 0, 1, 2$$

and for all $x \in [x_k, x_{k+1}]$, $k = 1(1)n-1$ we have:

$$|y^{(i)}(x) - S_k^{(i)}(x)| \leq c^* \omega(h), \quad i = 0, 1, 2$$

and

$$|z^{(i)}(x) - \bar{S}_k^{(i)}(x)| \leq K^* \omega(h), \quad i = 0, 1, 2$$

where c, K, c^* and K^* are constants independent of h.

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