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ON THE DYADIC DIFFERENTIABILITY OF DYADIC INTEGRAL FUNCTIONS ON $\mathbf{R}^+$

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Dedicated to Prof. I. Kátai
on the occasion of his fiftieth birthday

In their paper [1] P. L. Butzer and H. J. Wagner have introduced the concept of dyadic derivative for functions defined on the dyadic field $\mathbf{R}^+$. Furthermore, Wagner [5] has defined the notion of dyadic integral as the inverse of the dyadic derivative and investigated, among others, the strong dyadic differentiability of dyadic integrals. In this paper we shall prove some estimates from which the result of Wagner easily follows. Moreover, these estimates can be used also for the proof of the almost everywhere dyadic differentiability of dyadic integrals. We shall concern ourselves with this question in a forthcoming paper. In connection with this see also [2] and [3] .

Let $f : \mathbf{R}^+ \to \mathbf{C}$ be a function defined on the dyadic field and let for every $n \in \mathbf{N}$

$$d_n f := \frac{1}{2} \sum_{j=-n}^{n} 2^j (f - \tau_{2^{-(j+1)}} f) \tag{1}$$

be the nth dyadic "differential quotient" of f, where τ_h $(h \in \mathbf{R}^+)$ are the dyadic translation operators:

$$(\tau_h f)(x) := f(x \dot{+} h) \quad (x, h \in \mathbf{R}^+). \tag{2}$$

If for some point $x \in \mathbf{R}^+$ the limit

$$\lim_{n\to\infty}(d_n f)(x) =: f^{[1]}(x) \tag{3}$$

exists, then we say that f is dyadically differentiable at $x \in \mathbf{R}^+$ and $f^{[1]}(x)$ is the dyadic derivative of f at $x \in \mathbf{R}^+$. If $f \in L^1(\mathbf{R}^+)$ is an integrable function and there exists a function $g \in L^1(\mathbf{R}^+)$ for which

$$\lim_{n\to\infty} \| d_n f - g \|_1 = 0 \tag{4}$$

holds, then f is said to be strongly dyadically differentiable in $L^1(\mathbf{R})^+$ and $Df := g$ is the strong dyadic derivative of f.

Let $n \in \mathbf{N}$ and define the function W_n by its Walsh-Fourier transform $\widehat{W}_n$ as follows:

$$\widehat{W}_n(y) := \begin{cases} 0, & y \in [0, 2^{-n}) \\ \frac{1}{y}, & y \in [2^{-n}, +\infty)\ . \end{cases} \tag{5}$$

Wagner has proved (see [5]) that there exists uniquely a function $W_n \in L^1(\mathbf{R}^+)$ for which (5) holds; moreover,

$$W_n(x) = \lim_{k\to\infty} \int_{2^{-n}}^{2^k} \frac{1}{y} w_x(y)dy \quad (x \in \mathbf{R}^+) \tag{6}$$

and the limit can be taken either in the $L^1(\mathbf{R}^+)$-norm or in the pointwise sense. Here and in the sequel the symbols w_x $(x \in \mathbf{R}^+)$ denote the generalized Walsh functions.

In the following we introduce the inverse operation of the dyadic derivative by the following definition (see [5]): if for a function $f \in L^1(\mathbf{R}^+)$ there exists a function $g \in L^1(\mathbf{R}^+)$ such that

$$\lim_{n\to\infty} \| W_n * f - g \|_1 = 0, \tag{7}$$

then g is called the strong dyadic integral of f and is denoted by If ($*$ denotes dyadic convolution) .

For this notion of dyadic integral Wagner proved that the following assertions are equivalent for $f, g \in L^1(\mathbf{R}^+)$:

$$(8) \qquad \begin{array}{ll} i) & g = If, \\ ii) & \widehat{g}(y) = \begin{cases} 0, & y = 0 \\ \frac{1}{y}\widehat{f}(y), & y > 0 . \end{cases} \end{array}$$

In the following we investigate the strong dyadic differentiability of the dyadic integral $If \in L^1(\mathbf{R}^+)$ for $f \in L^1(\mathbf{R}^+)$. We remark that if $f \in L^1(\mathbf{R}^+)$ then If is not necessarily defined. For example, if $f := \chi[0,1)$ is the characteristic function of the interval $[0,1)$ then If is not defined (see [5]). Therefore, in the following we suppose that $f \in L^1(\mathbf{R}^+)$ and the dyadic integral $If \in L^1(\mathbf{R}^+)$ of f exists. First we compute the dyadic "differential quotients" $d_n(If)$ $(n \in \mathbf{N})$ of If . In connection with this we shall show the following

Lemma 1. *If for a function $f \in L^1(\mathbf{R}^+)$ the dyadic integral $If \in L^1(\mathbf{R}^+)$ exists, then*

$$(9) \qquad d_n(If) = d_nW_n * f \quad (n \in \mathbf{N}).$$

Proof. Since τ_h $(h \in \mathbf{R}^+)$ are isometries in $L^1(\mathbf{R}^+)$, from the definition of d_n it follows that

$$d_n(If) = \lim_{m\to\infty} d_n(W_m*f) = \lim_{m\to\infty}(d_nW_m)*f = (\lim_{m\to\infty} d_nW_m)*f,$$

where the limit is taken in the $L^1(\mathbf{R}^+)$-norm sense. Furthermore,

$$(10) \qquad d_nW_m = \lim_{k\to\infty} d_nW_{m,k}$$

and the limit can be taken either in the $L^1(\mathbf{R}^+)$-norm or in the pointwise sense, where

$$W_{m,k}(x) := \int_{2^{-m}}^{2^k} \frac{1}{y} w_x(y)dy \quad (x \in \mathbf{R}^+,\ m,k \in \mathbf{N}) . \tag{11}$$

For $m \geq n$ and $k \in \mathbf{N}$ we have

$$(d_n W_{m,k})(x) = \frac{1}{2} \sum_{j=-n}^{n} \int_{2^{-m}}^{2^k} \frac{2^j}{y} \left(w_x(y) - w_{x \dot{+} 2^{-(j+1)}}(y)\right) dy =$$

$$= \int_{2^{-m}}^{2^k} \frac{1}{y} \alpha_n(y) w_x(y) dy \quad (x \in \mathbf{R}^+),$$

where

$$\begin{gathered} \alpha_n(y) := \frac{1}{2} \sum_{j=-n}^{n} 2^j \left(1 - w_{2^{-(j+1)}}(y)\right) = \sum_{j=-n}^{n} y_j 2^{-j} \\ \left(y = \sum_{j=-\infty}^{\infty} y_j 2^{-j} \in \mathbf{R}^+,\ y_j \in \{0,1\},\ n \in \mathbf{N}\right). \end{gathered} \tag{12}$$

Since $\alpha_n(y) = 0$ if $y \in [0, 2^{-n})$, we have that for any $m \geq n$ and $k \in \mathbf{N}$

$$d_n W_{m,k} = d_n W_{n,k},$$

and, consequently, our lemma is proved. □

In the following our aim is to give an estimate for the functions $d_n W_n$ $(n \in \mathbf{N})$. To this end, we define $\beta_n(y)$ for $y \in \mathbf{R}^+$ and $n \in \mathbf{N}$ as follows:

$$\beta_n(y) := \sum_{j=-n}^{0} y_j 2^{-j}. \tag{13}$$

It is easy to see that

$$2^n \alpha_n(2^{-n}y) = \beta_{2n}(y) \quad (y \in \mathbf{R}^+,\ n \in \mathbf{N}). \tag{14}$$

Let us introduce the functions V_n $(n \in \mathbf{N})$ by the following equality:

$$V_n(x) := \lim_{k\to\infty} \int_0^{2^k} \frac{1}{y}\beta_n(y) w_x(y)\,dy \quad (x \in \mathbf{R}^+,\ n \in \mathbf{N}) \tag{15}$$

(it is easy to show that this limit exists). With the functions V_n $(n \in \mathbf{N})$ we can express the functions $d_n W_n$ $(n \in \mathbf{N})$. Namely, the following lemma is true.

Lemma 2. *For every $n \in \mathbf{N}$ and $x \in \mathbf{R}^+$*

$$(d_n W_n)(x) = 2^{-n} V_{2n}(2^{-n}x). \tag{16}$$

Proof. Using in the integral (15) the transformation $z := 2^{-n}y$ and (14), we get

$$(d_n W_n)(x) = \lim_{k\to\infty} \int_0^{2^k} \frac{1}{z}\alpha_n(z) w_x(z)\,dz =$$

$$= \lim_{k\to\infty} 2^{-n} \int_0^{2^{k+n}} \frac{1}{2^{-n}y}\alpha_n(2^{-n}y) w_x(2^{-n}y)\,dy =$$

$$= 2^{-n} \lim_{k\to\infty} \int_0^{2^k} \frac{1}{y}\beta_{2n}(y) w_{2^{-n}x}(y)\,dy =$$

$$= 2^{-n} V_{2n}(2^{-n}x) \quad (x \in \mathbf{R}^+,\ n \in \mathbf{N}). \ \square$$

In the following we give an estimate for the functions V_n $(n \in \mathbf{N})$. For this we define the functions L and J as follows:

$$L(u) := \begin{cases} 2^{-n-s-2}, & u \in [2^n + 2^s + k,\ 2^n + 2^s + k + 1) \\ & (0 \le k < 2^s,\ 0 \le s < n,\ n \in \mathbf{P}) \\ 0, & \text{otherwise,} \end{cases} \tag{17}$$

$$J(u) := \begin{cases} 1, & u \in [0,1) \\ 2^{-n-1}, & u \in [2^n,\ 2^n + 1),\ n \in \mathbf{N} \\ 0, & \text{otherwise.} \end{cases} \tag{18}$$

It is easy to see that $L, J \in L^1(\mathbf{R}^+)$ and

$$\| J \|_1 = 2, \qquad \| L \|_1 = \frac{1}{2}\,. \tag{19}$$

Using the functions L and J we can give an estimate for the functions V_n $(n \in \mathbf{N})$. For this we denote by $d_n W$ $(n \in \mathbf{N})$ the functions defined on the interval $[0,1)$ in [4]. In this case the following assertion is true.

Theorem 1. *For every $n \in \mathbf{N}$ we have the estimate*

$$| V_n | \le 7(L + J) + 2J \sum_{k=0}^{\infty} 2^{-k} \overline{D}_{2^k} + \chi[0,1)\,(| d_n W | + 1)\ , \tag{20}$$

where $\overline{D}_{2^k}$ $(k \in \mathbf{N})$ is the periodic extension of the Walsh-Dirichlet kernel D_{2^k} from the interval $[0,1)$ to $\mathbf{R}^+$ with period 1.

Proof. Since

$$\beta_n(y) = k \quad \left(y \in [i2^n + k, i2^n + k + 1)\right) \tag{21}$$

for $i, n \in \mathbf{N}$, $0 \le k < 2^n$, $V_n(x)$ can be written in the following form:

$$V_n(x) = \sum_{k=1}^{2^n - 1} w_k(x) \int_k^{k+1} \frac{k}{y} w_{[x]}(y)\,dy +$$

(22)

$$+ \sum_{i=1}^{\infty} \Big(\sum_{k=0}^{2^n - 1} w_{i2^n + k}(x) \int_{i2^n + k}^{i2^n + k + 1} \frac{k}{y} w_{[x]}(y)\,dy \Big)\ (x \in \mathbf{R}^+,\ n \in \mathbf{N}).$$

In the following we use the notation

$$(23)\quad A_{i2^n+k}(x) := \int_{i2^n+k}^{i2^n+k+1} \Big(\frac{k}{y} - \frac{k}{i2^n+k}\Big) w_{[x]}(y)\,dy \quad (x \in \mathbf{R}^+),$$

where for $i \in \mathbf{P}$, $n \in \mathbf{N}$ and $0 \le k < 2^n$ and for $i = 0,\ 1 \le k < 2^n$. Using this notation we have

$$(24)\quad \begin{aligned} V_n = \sum_{k=1}^{2^n-1} A_k w_k + \sum_{i=1}^{\infty} \Bigg(\sum_{k=0}^{2^n-1} A_{i2^n+k} w_{i2^n+k} \Bigg) + \\ + \chi[0,1)(d_n W - 1) \qquad (n \in \mathbf{N}). \end{aligned}$$

Let us introduce the following notations:

$$(25)\quad \begin{aligned} V_n^1 &:= \sum_{k=1}^{2^n-1} A_k w_k, \\ V_n^2 &:= \sum_{i=1}^{\infty} \Big(\sum_{k=0}^{2^n-1} A_{i2^n+k} w_{i2^n+k} \Big) \quad (n \in \mathbf{N}). \end{aligned}$$

Firstly we investigate the functions V_n^2 $(n \in \mathbf{N})$ and we shall show that

$$(26)\quad | V_n^2 | \le 3(L+J) \quad (n \in \mathbf{N}).$$

Using the second mean value theorem of integral calculus and integration by parts, we get that for every $x \in [1, +\infty)$, $i \in \mathbf{P}$ and $0 \le k < 2^n$ we have

$$(27)\quad \begin{aligned} A_{i2^n+k}(x) &= \int_{i2^n+k}^{i2^n+k+1} \frac{k}{y^2} J_{[x]}(y)\,dy = \\ &= \frac{k}{(i2^n+k)^2} \int_0^{\xi} J_{[x]}(y)\,dy + \frac{k}{(i2^n+k+1)^2} \int_{\xi}^{1} J_{[x]}(y)\,dy, \end{aligned}$$

where $\xi \in (0,1)$ is an appropriate number, and $J_{[x]}$ $(x \in \mathbf{R}^+)$ denotes the integral function of $w_{[x]}$ vanishing at 0 :

$$J_{[x]}(t) := \int_0^t w_{[x]}(u)du \quad (t, x \in \mathbf{R}^+). \tag{28}$$

In the following we shall need some estimates for the functions

$$L(x,\omega) := \int_0^\omega J_{[x]}(t)dt \quad (x, \omega \in \mathbf{R}^+). \tag{29}$$

It can be shown that

$$| J_{2^n+k}(t) | \leq 2^{-n-1} \ (t \in \mathbf{R}^+, \ 0 \leq k < 2^n, \ n \in \mathbf{N}) \tag{30}$$

and $\omega \to L(x,\omega)$ is a 1-periodic function, if $[x] \neq 2^l$ for some $l \in \mathbf{N}$ and $x \geq 1$.

Furthermore, it is easy to check that

$$\begin{gathered} | L(2^n + 2^s + k, \omega) | \leq 2^{-n-s-2} \\ (\omega \in \mathbf{R}^+, \ 0 \leq k < 2, \ 0 \leq s < n, \ n \in \mathbf{P}). \end{gathered} \tag{31}$$

Using the definition of the functions L and J (see(17), (18)), from (30) and (31) we get the following estimate:

$$| L(x,\omega) - L(x,[\omega]) | \leq L(x) + J(x) \quad (x, \omega \in \mathbf{R}^+). \tag{32}$$

From this and (27) the following estimate follows:

$$\begin{aligned} | A_{i2^n+k}(x) | &\leq \frac{k}{i^2 \cdot 2^{2n}} (2 \mid L(x,\xi) \mid + \mid L(x,1) \mid) \leq \\ &\leq \frac{3k}{i^2 \cdot 2^{2n}} (L(x) + J(x)) \quad (x \in [1, +\infty)). \end{aligned} \tag{33}$$

Using this, we get that on the interval $[1,+\infty)$

$$(34)\qquad \begin{aligned} \mid V_n^2 \mid &\le \sum_{i=1}^{\infty}\Big(\sum_{k=0}^{2^n-1} \mid A_{i2^n+k} \mid\Big) \le \\ &\le \frac{3}{2}(L+J)\sum_{i=1}^{\infty}\frac{1}{i^2} < 3(L+J) \quad (n \in \mathbf{N}). \end{aligned}$$

For $x \in [0,1)$, $i \in \mathbf{P}$ and $0 \le k < 2^n$ we have that

$$(35)\qquad A_{i2^n+k}(x) = k \cdot \left[\ln\left(1+\frac{1}{i2^n+k}\right) - \frac{1}{i2^n+k}\right].$$

From this using the inequality

$$\mid \ln(1+t) - t \mid \le \frac{1}{2}t^2 \ (0 \le t < 1),$$

it follows that on the interval $[0,1)$

$$(36)\quad \mid A_{i2^n+k} \mid \le \frac{k}{2(i2^n+k)^2} \le \frac{k}{2i^2\cdot 2^{2n}} \quad (i \in \mathbf{P},\ 0 \le k < 2^n)$$

and, since $L=0$ and $J=1$ on the interval $[0,1)$, we get that on the interval $[0,1)$

$$(37)\ \mid V_n^2 \mid \le \sum_{i=1}^{\infty}\Big(\sum_{k=0}^{2^n-1} \mid A_{i2^n+k} \mid\Big) \le \frac{1}{2} < 3(L+J) \quad (n \in \mathbf{N}).$$

Consequently, for the functions V_n^2 $(n \in \mathbf{N})$ we have the estimate (26).

In the following we investigate the functions V_n^1 $(n \in \mathbf{N})$. Namely, we shall show that

$$(38)\qquad \mid V_n^1 \mid \le 4(L+J) + 2J\Big(\sum_{k=0}^{\infty} 2^{-k}\overline{D}_{2^k}\Big)\ (n \in \mathbf{N}).$$

For $1 \leq k < 2^n$ and $x \in [1,+\infty)$, using integration by parts, we get that

$$A_k(x) = \int_k^{k+1} (\frac{k}{y} - 1) w_{[x]}(y) dy =$$

$$(39) \quad = \int_0^1 \left(\frac{1}{1+\frac{y}{k}-1} \right) w_{[x]}(y) dy = \frac{1}{k} \int_0^1 \frac{1}{(1+\frac{y}{k})^2} J_{[x]}(y) dy =$$

$$= \frac{1}{k} \int_0^1 \left(\frac{1}{(1+\frac{y}{k})^2} - 1 \right) J_{[x]}(y) dy + \frac{1}{k} \int_0^1 J_{[x]}(y) dy :=$$

$$:= A_k^1(x) + A_k^2(x).$$

Using the second mean value theorem of integral calculus, we have

$$(40) \quad A_k^1(x) = \frac{1}{k} \left(\frac{1}{(1+\frac{1}{k})^2} - 1 \right) \cdot \int_\xi^1 J_{[x]}(y) dy$$

$$(x \in [1,+\infty),\ 1 \leq k < 2^n)$$

with an appropriate number $\xi \in (0,1)$. Using this,we get the following estimate on the interval $[1,+\infty)$:

$$(41) \quad \left| \sum_{k=1}^{2^n-1} A_k^1 w_k \right| \leq \sum_{k=1}^{2^n-1} | A_k^1 | \leq$$

$$\leq \sum_{k=1}^{2^n-1} \frac{2}{k^2}(L+J) \leq 4(L+J) \quad (n \in \mathbf{N}).$$

From the definition of A_k^2 we have

$$(42) \quad \sum_{k=1}^{2^n-1} A_k(x) w_k(x) = \Big(\sum_{k=1}^{2^n-1} \frac{w_k(x)}{k} \Big) \cdot \int_0^1 J_{[x]}(y) dy$$

$$(x \in [1,+\infty),\ n \in \mathbf{N}).$$

By the definition of J and an estimate for the first factor in (42) (see [4]), we get

$$
\begin{aligned}
&\left| \sum_{k=1}^{2^n-1} A_k^2(x) w_k(x) \right| \leq 4 \left(\sum_{k=0}^{\infty} 2^{-k} \overline{D}_{2^k}(x) \right) \cdot \int_0^1 J_{[x]}(y)\,dy = \\
(43) \qquad & \\
&= 2J(x) \left(\sum_{k=0}^{\infty} 2^{-k} \overline{D}_{2^k}(x) \right) \quad (x \in [1, +\infty),\ n \in \mathbf{N}).
\end{aligned}
$$

For $x \in [0,1)$ and $1 \leq k < 2^n$ we can calculate $A_k(x)$ simply as follows:

$$
(44) \qquad A_k(x) = \int_k^{k+1} \left(\frac{k}{y} - 1\right) dy = k \ln\left(1 + \frac{1}{k}\right) - 1.
$$

Write the sum

$$
\sum_{k=1}^{2^n-1} A_k w_k
$$

on the interval $[0,1)$ in the form

$$
\begin{aligned}
&\sum_{k=1}^{2^n-1} A_k w_k = \\
(45) \qquad & \\
&= \sum_{k=1}^{2^n-1} \left(k \ln\left(1 + \frac{1}{k}\right) - 1 + \frac{1}{2k}\right) w_k - \frac{1}{2} \sum_{k=1}^{2^n-1} \frac{w_k}{k} \quad (n \in \mathbf{N}).
\end{aligned}
$$

Since

$$
\left| \ln(1+t) - t + \frac{t^2}{2} \right| \leq \frac{1}{3} t^3 \quad (0 \leq t < 1),
$$

from (45) we get the following estimate on the interval $[0,1)$:

$$(46)\qquad \begin{aligned}|\sum_{k=1}^{2^n-1} A_k w_k| &\le \frac{1}{3}\sum_{k=1}^{2^n-1}\frac{1}{k^2}+\frac{1}{2}|\sum_{k=1}^{2^n-1}\frac{w_k}{k}|\le \\ &\le \frac{2}{3}+2\sum_{k=0}^{\infty}2^{-k}\overline{D}_{2^k} \quad (n\in \mathbf{N}).\end{aligned}$$

Summarizing our results, we get the desired estimate (38) for the functions V_n^1 $(n \in \mathbf{N})$ and on the basis of (24), (26) and (38) we have proved the theorem. □

In the following we shall show that the estimate (20) for V_n $(n \in \mathbf{N})$ implies among others the strong dyadic differentiability of the dyadic integrals (see also [5]). Namely, the following theorem is true.

Theorem 2. i) $\sup\limits_{n\in\mathbf{N}} \| V_n \|_1 < +\infty$.

ii) *If for a function* $f \in L^1(\mathbf{R}^+)$

$$(47)\qquad \hat{f}(0) = \int_{\mathbf{R}^+} f = 0$$

and the dyadic integral $If \in L^1(\mathbf{R}^+)$ *exists, then* If *is strongly dyadically differentiable and*

$$(48)\qquad D(If) = f.$$

Proof. i) We know that $L, J \in L^1(\mathbf{R}^+)$ and on the basis of the definition of J we have

$$(49)\qquad \begin{aligned}&\int_{\mathbf{R}^+} J\left(\sum_{k=0}^{\infty}2^{-k}\overline{D}_{2^k}\right) = \\ &= \int_0^1 \left(\sum_{k=0}^{\infty}2^{-k}\overline{D}_{2^k}\right) + \sum_{j=0}^{\infty}2^{-j-1}\int_{2^j}^{2^j+1}\left(\sum_{k=0}^{\infty}2^{-k}\overline{D}_{2^k}\right) \le 4.\end{aligned}$$

Furthermore, using a result in [4], one has

$$\sup_{n\in\mathbf{N}} \int_0^1 | d_n W | < +\infty. \tag{50}$$

Thus, by the estimate (20) for V_n $(n \in \mathbf{N})$ part i) of the theorem is proved.

ii) Define the operators T_n $(n \in \mathbf{N})$ on $L^1(\mathbf{R}^+)$ as follows:

$$T_n f := d_n W_n * f \quad (f \in L^1(\mathbf{R}^+),\ n \in \mathbf{N}). \tag{51}$$

The operators T_n $(n \in \mathbf{N})$ are uniformly bounded. In fact, by (16) we have

$$\begin{aligned} \| T_n \| \le \| d_n W_n \|_1 &= \int_{\mathbf{R}^+} 2^{-n} \mid V_{2n}(2^{-n}x) \mid dx = \\ &= \| V_{2n} \|_1 = O(1) \quad (n \to \infty). \end{aligned} \tag{52}$$

Thus, by (9) and Banach–Steinhaus theorem it is enough to show (48) for the elements of a dense subset in the dyadically integrable functions satisfying (47). To this end, let us define the set of functions

$$S := \{\chi[0, 2^s) w_{2^{-s}m} :\ s \in \mathbf{N},\ m \in \mathbf{P}\}. \tag{53}$$

Then it can be shown that S is a dense subset in the set of dyadically integrable functions.

Firstly we shall show that if $f \in L^1(\mathbf{R}^+)$ is a dyadically integrable function, then

$$\lim_{n\to\infty} 2^n \int_0^{2^n} f = 0. \tag{54}$$

Let $h := \chi[0,2^{-n})$ $(n \in \mathbf{N})$. Then its Walsh–Fourier transform is

$$\widehat{h} = 2^{-n}\chi[0,2^n)$$

and by the equality

$$\int\limits_{\mathbf{R}^+} f\widehat{h} = \int\limits_{\mathbf{R}^+} \widehat{f}h$$

we get

$$2^{-n}\int\limits_0^{2^n} f = \int\limits_0^{2^{-n}} \widehat{f}. \tag{55}$$

Putting $g := If$, by (8) we have

$$\int\limits_0^{2^{-n}} \widehat{f} = \int\limits_0^{2^{-n}} y\widehat{g}(y)dy. \tag{56}$$

Since $\widehat{g}(0) = 0$ and $\widehat{g}$ is w-continuous, we can estimate $\widehat{g}$ by its modulus of continuity as follows:

$$| \widehat{g}(y) | = | \widehat{g}(y) - \widehat{g}(0) | \leq \omega(\widehat{g}, 2^{-n}) \quad (y \in [0, 2^{-n})). \tag{57}$$

From this we get

$$\left| \int\limits_0^{2^{-n}} y\widehat{g}(y)\, dy \right| \leq \omega(\widehat{g}, 2^{-n}) \int\limits_0^{2^{-n}} y\, dy = \frac{1}{2}2^{-2n}\omega(\widehat{g}, 2^{-n}),$$

and, consequently, by (55) and (56) the inequality (54) follows.

In the following let $f \in L^1(\mathbf{R}^+)$ be a dyadically integrable function and ε an arbitrary positive number. Let us decompose the function f in the form

$$f = \chi[0,2^N)f + \chi[2^N, +\infty)f := f_1 + f_2, \tag{58}$$

where $N \in \mathbf{N}$ is natural number for which

(59) $$\| f_2 \|_1 = \int_{2^N}^{\infty} | f | < \varepsilon, \quad 2^N \, | \int_0^{2^N} f | < \varepsilon$$

hold. Let us define an element P in the linear hull of S as follows:

(60) $$P := S_{2^n} f_1 - (\int_0^{2^N} f) \chi[0, 2^N),$$

where $S_{2^N} f_1$ denotes the 2^N-th partial sum of f_1 on the interval $[0, 2^N)$. Then

(61) $$\lim_{N \to \infty} \| f_1 - S_{2^N} f_1 \|_1 = 0$$

and, consequently, by (59) and (61)

(62) $$\begin{aligned} &\| f - P \|_1 \leq \| f_1 - P \|_1 + \| f_2 \|_1 \leq \\ &\leq \| f_1 - S_{2^N} f_1 \|_1 + \| (\int_0^{2^N} f) \chi[0, 2^N) \|_1 + \| f_2 \|_1 < 3\varepsilon, \end{aligned}$$

if N is large enough. Thus we have proved that S is a dense subset in the set of dyadically integrable functions.

In the following we shall show that for elements of S (48) is true. Let

(63) $$f := \chi[0, 2^s) w_{2^{-s} m} \quad (s \in \mathbf{N}, \ m \in \mathbf{P})$$

be an arbitrary element of S. We must show that

(64) $$\lim_{n \to \infty} \| d_n W_n * f - f \|_1 = 0.$$

By an easy calculation we get that

$$(65)\qquad \begin{aligned}&(d_nW_n * f)(x) - f(x) = \\ &= \int\limits_{\mathbf{R}^+} \frac{\alpha_n(y) - y}{y}\widehat{f}(y)w_x(y)dy \quad (x \in \mathbf{R}^+,\ n \in \mathbf{N})\end{aligned}$$

and

$$(66)\qquad \widehat{f} = 2^s\chi[\frac{m}{2^s}, \frac{m+1}{2^s}).$$

Consequently,

$$(67)\qquad \begin{aligned}&(d_nW_n * f)\,(x) - f(x) = \\ &= 2^s \int\limits_{m/2^s}^{(m+1)/2^s} \frac{\alpha_n(y) - y}{y}w_x(y)dy \quad (x \in \mathbf{R}^+,\ n \in \mathbf{N}).\end{aligned}$$

Let us introduce the following notation:

$$F_n(x) := (d_nW_n * f)(2^sx) - f(2^sx) \quad (x \in \mathbf{R}^+,\ n \in \mathbf{N}).$$

It is enough to show that

$$(69)\qquad \lim_{n\to\infty} \| F_n \|_1 = 0.$$

Applying a simple transformation we can show that

$$(70)\qquad F_n(x) = 2^sw_m(x)\int\limits_0^1 \gamma_n(u)\gamma(u)w_{[x]}(u)du \quad (x \in \mathbf{R}^+,\ n \in \mathbf{N}),$$

where

$$\gamma_n(u) := \alpha_n(2^{-s}u) - 2^{-s}u,$$
$$(71) \qquad \gamma(u) := \frac{1}{m+u} \quad (0 \le u < 1).$$

From (70) we get that

$$(72) \qquad \begin{aligned} &\| F_n \|_1 = \| (\gamma_n \gamma)^\wedge \|_{l^1} = \\ &= \| \widehat{\gamma}_n * \widehat{\gamma} \|_{l^1} \le \| \widehat{\gamma}_n \|_{l^1} \| \widehat{\gamma} \|_{l^1} \quad (n \in \mathbf{N}), \end{aligned}$$

where $\widehat{\varphi}$ denotes the sequence of Walsh–Fourier coefficients for any function φ defined $[0,1)$. Integrating by parts one can show that

$$(73) \qquad \| \widehat{\gamma} \|_{l^1} < +\infty.$$

Moreover, since

$$(74) \quad \gamma_n(u) = \sum_{k=n+1}^{\infty} \frac{w_{2^{k-s}}(u)}{2^{k+1}} - \frac{1}{2^{n+1}} \quad (0 \le u < 1,\ n \in \mathbf{N}),$$

we get

$$(75) \qquad \| \gamma_n \|_{l^1} = \sum_{k=n+1}^{\infty} \frac{1}{2^{k+1}} + \frac{1}{2^{n+1}} = \frac{1}{2^n} \quad (n \in \mathbf{N}).$$

Consequently, by (72) we have

$$\lim_{n\to\infty} \| F_n \|_1 = 0. \ \square$$

References

[1] BUTZER, P. L. and WAGNER, H. J., A calculus for Walsh functions defined on $\mathbf{R}^+$. In: *Proc. Symp. on Applications of Walsh Functions.* Washington D. C., 1973, 75-81.

[2] PÁL, J. On almost everywhere differentiability of dyadic integral functions on $\mathbf{R}^+$. In: *Colloq. Math. Soc. J. Bolyai, 19. Fourier Analysis and Approximation Theory*, Vol. II. Budapest, 1978, 591-601.

[3] PÁL, J., The almost everywhere differentiability of Wagner's dyadic integral function on $\mathbf{R}^+$. In: *Colloq. Math. Soc. J. Bolyai, 49. Alfred Haar Memorial Conference.* Budapest, 1985, 695-701.

[4] SCHIPP, F., Über einen Ableitungsbegriff von P. L. Butzer und H. J. Wagner. *Math. Balkanica* 4 (1974) 541-546.

[5] WAGNER, H. J., On the dyadic calculus for functions defined on $\mathbf{R}^+$. In: *Theory and Applications of Walsh Functions and Other Nonsinusoidal Functions.* Hatfield Polytechnic, Hatfield, 1975, 101-129.

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