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FUNCTIONAL EQUATIONS OF SUM TYPE ON A DOMAIN

ZOLTÁN DARÓCZY and LÁSZLÓ LOSONCZI

Dedicated to Professor I. Kátai on his 50^{th} birthday

1. Introduction

Let $n \geq 3$ be a natural number and $S \subset \mathbf{R}^n$ a nonempty set. The function $f : \mathbf{R} \longrightarrow \mathbf{R}$ is said to satisfy *a functional equation of sum type* on the set S if for every $x \in S$

$$\sum_{i=1}^{n} f(P_i x) = 0 \tag{1.1}$$

holds, where $P_i x = P_i(x_1, x_2, \ldots, x_n) := x_i \quad (i = 1, 2, \ldots, n)$.

G. Székely [4] proposed the investigation of the following case. Let $a = (a_1, a_2, \ldots, a_n) \in \mathbf{R}^n$ be a given vector with non-negative coordinates and define S by

$$S = S(a) := \{x \in \mathbf{R}^n \mid P_i x \leq P_{i+1} x \quad (i = 1, 2, \ldots, n-1), \ \langle a, x \rangle = 0\}, \tag{1.2}$$

where

$$\langle a, x \rangle = \sum_{i=1}^{n} a_i x_i$$

is the inner product of a and x.

Supposing the measurability of f Székely solved (1.1) on $S(a)$. Here we find the general solution of (1.1) on $S(a)$.

2. Results

We have to distinguish the following two cases:

(*i*) $a_i > 0$ for all $i = 1, 2, \dots, n$;

(*ii*) there exists a $j \in \{1, 2, \dots, n\}$ such that $a_j = 0$.

Theorem 1. *Let $n \geq 3$ and $a = (a_1, a_2, \dots, a_n)$ be a positive vector i.e. $a_i > 0$ $(i = 1, 2, \dots, n)$. If $f : \mathbf{R} \longrightarrow \mathbf{R}$ is a non identically zero function satisfying the functional equation (1.1) for all $x \in S(a)$, then*

$$a_1 = a_2 = \dots = a_n \tag{2.1}$$

and there exists a non identically zero additive function $A : \mathbf{R} \longrightarrow \mathbf{R}$ such that

$$f(x) = A(x) \qquad (x \in \mathbf{R}). \tag{2.2}$$

Theorem 2. *Let $n \geq 3$ and $a = (a_1, a_2, \dots, a_n)$ be a vector with nonnegative coordinates such that at least one of the coordinates is zero. If $f : \mathbf{R} \longrightarrow \mathbf{R}$ satisfies (1.1) for all $x \in S(a)$, then*

$$f(x) = 0 \qquad (x \in \mathbf{R}). \tag{2.3}$$

Summarizing these theorems we can state the following.

Suppose that $n \geq 3$ and $a = (a_1, a_2, \dots, a_n)$ has nonnegative coordinates. The functional equation (1.1) on the set $S(a)$ has non identically zero solution $f : \mathbf{R} \longrightarrow \mathbf{R}$ if and only if

$$a_1 = a_2 = \dots = a_n > 0 \tag{2.4}$$

holds and then the general solution is a nonzero additive function.

COROLLARY. Let $n \geq 3$ and $a = (a_1, a_2, \ldots, a_n)$ be a given vector with nonnegative coordinates. Suppose that $f : \mathbf{R} \longrightarrow \mathbf{R}$ satisfies one of the following conditions:

(j) f is continuous at a point,

(jj) f is nonnegative for small positive x,

(jjj) f is bounded in an interval,

(jv) f is measurable,

(v) f is majorizable (or minorizable) through a measurable function on a set of positive measure.

If f satisfies (1.1) on $S(a)$, then

$$f(x) = cx \qquad (x \in \mathbf{R}),$$

where c is a constant. This function satisfies (1.1) either if $c = 0$ or if $c \neq 0$ and (2.4) holds.

3. Proofs

Proof of Theorem 1. Let $f \neq 0$ be a solution of (1.1) on $S(a)$ and let

$$\triangle = \left\{ (x, y) \mid x \geq 0,\ y \geq 0,\ \frac{x}{a_{n-1}} \leq \frac{y}{a_n} \right\}.$$

With $x_1 = x_2 = \ldots = x_n = 0$ we obtain from (1.1) that $f(0) = 0$.

Substituting

$$x_1 = -\frac{x+y}{a_1},\ x_2 = \cdots = x_{n-2} = 0,\ x_{n-1} = \frac{x}{a_{n-1}},\ x_n = \frac{y}{a_n}$$

into (1.1) we get

$$f\left(-\frac{x+y}{a_1}\right) + f\left(\frac{x}{a_{n-1}}\right) + f\left(\frac{y}{a_n}\right) = 0. \tag{3.1}$$

With $x = 0$, $y \geq 0$ we have

$$f\left(\frac{y}{a_n}\right) = -f\left(-\frac{y}{a_1}\right), \tag{3.2}$$

therefore for any $x \geq 0$

$$f\left(\frac{x}{a_{n-1}}\right) = f\left(\frac{xa_n}{a_{n-1}a_n}\right) = -f\left(-\frac{xa_n}{a_{n-1}a_1}\right).$$

Thus (3.1) can be written as

$$f\left(-\frac{x+y}{a_1}\right) = f\left(\frac{-xa_n}{a_{n-1}a_1}\right) + f\left(-\frac{y}{a_1}\right) \quad ((x,y) \in \triangle). \tag{3.3}$$

Introducing the notation

$$g(s) := f\left(-\frac{s}{a_1}\right) \qquad (s \geq 0)$$

we have

$$g(x+y) = g(\alpha x) + g(y) \quad \big((x,y) \in \triangle\big), \tag{3.4}$$

where $\alpha := a_n/a_{n-1} > 0$.

For arbitrary $x \geq 0$, $y \geq 0$ choose $z > 0$ such that

$$\frac{x+y}{a_{n-1}} \leq \frac{z}{a_n},$$

then

$$\frac{x}{a_{n-1}} \leq \frac{x+y}{a_{n-1}} \leq \frac{z}{a_n} \leq \frac{y+z}{a_n}, \quad \frac{y}{a_{n-1}} \leq \frac{x+y}{a_{n-1}} \leq \frac{z}{a_n} \tag{3.5}$$

holds. (3.5) shows that $(x, y+z) \in \triangle$, $(y,z) \in \triangle$ and $(x+y, z) \in \triangle$.
By (3.4) we obtain that

$$g\left[x + (y+z)\right] = g(\alpha x) + g(y+z) = g(\alpha x) + g(\alpha y) + g(z)$$

and

$$g\left[(x+y)+z\right] = g\big(\alpha(x+y)\big) + g(z)$$

holds, therefore

$$g\big(\alpha(x+y)\big) = g(\alpha x) + g(\alpha y)$$

for every $x \geq 0$, $y \geq 0$. Thus the function $s \to g(\alpha s)$ is additive on the set $\{(x,y) \mid x \geq 0, y \geq 0\}$. By the extension theorem of additive functions (see J. Aczél and P. Erdős [2], Z. Daróczy and L. Losonczi [3]) there exists an additive function $\bar{A} : \mathbf{R} \longrightarrow \mathbf{R}$ such that

$$g(\alpha s) = \bar{A}(s) \quad \text{if } s \geq 0,$$

or

$$(3.6) \quad f(x) = g(-a_1 x) = \bar{A}(-\frac{a_1 x}{\alpha}) = -\bar{A}(\frac{a_1 x}{\alpha}) \qquad \text{if } x \leq 0.$$

On the basis of (3.2), (3.6)

$$f(x) = -f\left(-\frac{a_n x}{a_1}\right) = -\bar{A}\left(\frac{a_n x}{\alpha}\right) \quad \text{if } x \geq 0.$$

With the notation

$$A(x) := -\bar{A}(\frac{a_1 x}{\alpha}) \quad \text{for} \quad x \in \mathbf{R}$$

we obtain that $A : \mathbf{R} \longrightarrow \mathbf{R}$ is an additive function and

$$(3.7) \qquad f(x) = \begin{cases} A(x) & \text{if } x \leq 0 \\ A(\frac{a_n x}{a_1}) & \text{if } x \geq 0. \end{cases}$$

Now we show that $a_1 = a_2 = \cdots = a_n$.
For this purpose choose $x > 0$, $y > 0$ such that

$$\frac{x}{a_{n-1} + a_{n-2} + \cdots + a_{n-k}} \leq \frac{y}{a_n} \qquad (1 \leq k \leq n-2)$$

holds and substitute

$$x_1 = -\frac{x+y}{a_1}, \quad x_2 = \ldots = x_{n-k-1} = 0,$$

$$x_{n-k} = \ldots = x_{n-1} = \frac{x}{a_{n-1} + \cdots + a_{n-k}}, \quad x_n = \frac{y}{a_n}$$

into (1.1). Using (3.7) we obtain that

$$A\left(-\frac{x+y}{a_1}\right) + kA\left(\frac{a_n}{a_1}\frac{x}{a_{n-1} + \cdots + a_{n-k}}\right) + A\left(\frac{a_n}{a_1}\frac{y}{a_n}\right) = 0.$$

Hence, by the additivity of A,

$$A\left[\left(\frac{ka_n}{a_{n-1} + \cdots + a_{n-k}} - 1\right)\frac{x}{a_1}\right] = 0 \tag{3.8}$$

for every $x > 0$. $f \neq 0$ implies that $A \neq 0$ thus (3.8) holds if and only if

$$ka_n = a_{n-1} + \cdots + a_{n-k} \qquad (1 \leq k \leq n-2).$$

Choosing $k = 1$ here we have $a_n = a_{n-1}$. With $k = 2$ we get $2a_n = a_{n-1} + a_{n-2}$ i.e. $a_n = a_{n-2}$. Continuing similarly we obtain that

$$a_n = a_{n-1} = \cdots = a_2. \tag{3.9}$$

Next we show that $a_2 = a_1$. Let $x > 0$ and substitute

$$x_1 = x_2 = -x, \quad x_3 = \cdots = x_n = \frac{a_1 + a_2}{a_3 + \cdots + a_n}x = \frac{a_1 + a_2}{(n-2)a_n}x$$

into (1.1). By (3.7) we have

$$A(-x) + A(-x) + (n-2)A\left(\frac{a_n}{a_1}\frac{a_1 + a_2}{(n-2)a_n}x\right) = 0,$$

therefore

$$A\left[\left(\frac{a_1+a_2}{a_1}-2\right)x\right]=0$$

for all $x > 0$. Since $A \neq 0$ we get

$$\frac{a_1+a_2}{a_1}-2=0,$$

i.e. $a_2 = a_1$. Thus (2.1) is proved and (2.2) is obtained from (3.7) by (2.1). □

Proof of Theorem 2. If all coordinates $a_1, a_2, \ldots, a_n$ are zeros, then by the substitution $x_1 = \cdots = x_{n-1} = 0$, $x_n \geq 0$ into (1.1) we get $f(x_n) = -(n-1)f(0) = 0$ $(x_n \geq 0)$. Substituting $x_1 \leq 0$, $x_2 = \cdots = x_n = 0$ into (1.1) we have $f(x_1) = 0$ if $x_1 \leq 0$, therefore (2.3) holds.

If among the coordinates $a_1, a_2, \ldots, a_n$ there is at least one zero and at least one positive element then we shall distinguish 3 cases.

Case 1: $a_1 = 0$. Let $k > 1$ such that $a_1 = \ldots = a_{k-1} = 0$ but $a_k > 0$. The substitution of $x_1 \leq 0$, $x_2 = \cdots = x_n = 0$ into (1.1) gives that $f(x_1) = 0$ for $x_1 \leq 0$.

If $a_i = 0$ for all i different from k, then putting $x_k \geq 0$, $x_i = 0$ if $i \neq k$ into (1.1) we get $f(x_k) = 0$ for $x_k \geq 0$ thus (2.3) holds.

If there is a subscript l ($k < l \leq n$) such that $a_l > 0$, then $a_{k+1} + \cdots + a_n > 0$ and we may substitute

$$x_1 = \cdots = x_k = \frac{x}{a_k} \qquad (x \leq 0),$$

$$x_{k+1} = \cdots = x_n = y = -\frac{x}{a_{k+1} + \cdots + a_n}$$

into (1.1). We get $(n-k)f(y) = 0$, $f(y) = 0$ for $y \geq 0$ thus (2.3) holds again.

Case 2: $a_n = 0$. Choose $k < n$ such that $a_n = a_{n-1} = \cdots = a_{k+1} = 0$ but $a_k > 0$. With $x_1 = \cdots = x_{n-1} = 0$, $x_n \geq 0$ we obtain from (1.1) that $f(x_n) = 0$ for $x_n \geq 0$.

If $a_i = 0$ for all i different from k, then the substitution $x_k \leq 0$, $x_i = 0$ for $i \neq k$ gives that $f(x_k) = 0$ for $x_k \leq 0$ hence (2.3) is valid.

If there is a subscript l $(1 \leq l < k)$ such that $a_l > 0$, then $a_1 + \cdots + a_{k-1} > 0$ and we may substitute (with $x \geq 0$)

$$x_1 = \cdots = x_{k-1} = y = -\frac{x}{a_1 + \cdots + a_{k-1}},$$
$$x_k = \cdots = x_n = \frac{x}{a_k}$$

into (1.1). We obtain $(k-1)f(y) = 0$, $f(y) = 0$ for $y \leq 0$ i.e. (2.3) holds again.

Case 3: $a_1 \neq 0$ and $a_n \neq 0$. There must be a subscript j $(1 < j < n)$ such that $a_j = 0$. Let β be an arbitrary positive number and $\alpha < 0$ be such that

$$\alpha' = -\frac{a_1 + \cdots + a_{j-1}}{a_{j+1} + \cdots + a_n}\alpha \geq \beta.$$

The substitution of the values

$$x_1 = \cdots = x_{j-1} = \alpha,\ x_j \in [0, \beta], \quad x_{j+1} = \cdots = x_n = \alpha'$$

into (1.1) gives the equation

$$(j-1)f(\alpha) + f(x_j) + (n-j)f(\alpha') = 0.$$

Since this holds for every $x_j \in [0, \beta]$, f must be constant on $[0, \beta]$. $\beta > 0$ is arbitrary and $f(0) = 0$, hence $f(x) = 0$ for all $x \geq 0$. Substituting

$$x_1 < 0,\ x_2 = \cdots = x_n = -\frac{a_1 x}{a_2 + \cdots + a_n} \geq 0$$

into (1.1) we get $f(x_1) = 0$ $(x_1 < 0)$ which shows that (2.3) holds again. □

PROOF OF THE COROLLARY. By Theorems 1 and 2 f is either the zero function or a nonzero additive function. If one of (j)–(v) holds, then it is well-known (see e.g. J. Aczél [1]) that this additive function is of the form $A(x) = cx$, c being a constant. □

References

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[2] ACZÉL, J. and ERDŐS, P., The non-existence of a Hamel-basis and the general solution of Cauchy's functional equation for nonnegative numbers. *Publ. Math. Debrecen* **12** (1965) 259-265.

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ZOLTÁN DARÓCZY LÁSZLÓ LOSONCZI

Dept. of Mathematics
Kossuth Lajos University
H-4010 Debrecen, Egyetem tér 1-3.

HUNGARY