REMARKS ON THE FEASIBILITY PROBLEM OF ORIENTED MATROIDS

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Abstract. This paper presents a new approach to the feasibility problem of oriented matroids. The distinguished element $e \in E$ is alternatively a basic and nonbasic element during the algorithm. This approach simplifies the pivoting rule.

The connection between this algorithm and the known ones (Bland [1], Terlaky [6]) is presented in the final part.

1. Introduction

The definition and the fundamental properties of oriented matroids are presented in the papers of Bland [1], Folkman and Lawrence [4], Bland and Las Vergnas [3]. We will use them without formulating again. The notations of Bland are used in this paper. The tableau construction of Bland [1] and the pivot transformation play an important role in our further considerations.

Now we present the feasibility problem of oriented matroids. Let $M = (E, \sigma)$ and $M^* = (E, \sigma^*)$ be dual pairs of oriented matroids. Let $e_1 \in E$ be the distinguished element, where $E = \{e_1, \dots, e_n\}$ is a finite set.

Definition 1.1. The oriented circuit $X = (X^+, X^-) \in \sigma$ is called *feasible*, if $e_1 \in X^+$ and $X^- = \emptyset$.

The same way, the oriented cocircuit $Y = (Y^+, Y^-) \in \sigma^*$ is called *dual feasible*, if $e_1 \in Y^+$ and $Y^- = \emptyset$.

It follows from the orthogonality property of oriented circuits and cocircuits (Bland [1]) that exactly one of feasible circuits of cocircuits exists. Now we give a new algorithmic proof of this fact, i. e. we generate a feasible circuit or cocircuit.

2. The finite algorithm

This algorithm is a very simple variant of Bland's [1] minimal subscript rule. The linear algebraic variant of this algorithm and its consequences are presented in the paper of Klafszky-Terlaky [5].

Let us start with a base B and the corresponding *tableau* $T(B)$. It is well-known that an oriented cocircuit Y and an oriented circuit X is associated with every row and column of a basic tableau respectively. The element $\tau_{ij} \in \in \{0, +1, -1\}$ of the tableau shows the sign of the element $e_j \in B$ in the oriented cocircuit Y_i associated with the basic element e_i ($\tau_{ii} = +1$). In terms of this notations the problem is finding a base B with the property $e_1 \notin B$ such that the column of e_1 is nonpositive (in this case $(-X_1)$ is a primal feasible circuit) or if $e_1 \in B$ and its row is nonnegative (in this case Y_1 is a dual feasible cocircuit).

Pivoting rule (P)

Step 0. Let a base B and the corresponding tableau $T(B)$ be given. If $e_1 \notin B$, then go to Step 1, if $e_1 \in B$ then go to Step 2.

Step 1. *a)* If $e_1 \notin B$ and $\tau_{i1} \leqq 0$ for all $e_i \in B$, then $(-X_1)$ is a primal feasible circuit. STOP.

b) If $e_1 \notin B$ and *a)* does not hold, then let $r = \min \{i \mid \tau_{i1} > 0 \text{ for } e_i \in B\}$. Make a pivot operation. e_1 enters and e_r leaves the base. Continue with Step 2.

Step 2. *a)* If $e_1 \in B$ and $\tau_{1j} \geqq 0$ for all $e_j \notin B$, then $(-Y_1)$ is a dual feasible cocircuit. STOP.

b) If $e_1 \in B$ and *a)* does not hold, then let $k = \min \{j \mid \tau_{1j} < 0 \text{ for } e_j \notin B\}$. Make a pivot operation. e_k enters and e_1 leaves the base. Go back to Step 1.

This algorithm results in either a primal or dual feasible oriented circuit or cocircuit, so we have to prove only that this algorithm cannot cycle. The novelty of this approach is that the element e_1 is alternatively a basic and a nonbasic element through this procedure, contrary to the known methods, where the place of the distinguished element e_1 is fixed.

The following theorem proves the finiteness of the procedure.

Theorem 2.1. *The algorithm defined by pivoting rule (P) is finite, that is cycling cannot occure.*

Proof. Let us suppose to the contrary that the algorithm cycles, that is starting from a base B we obtain again the base B. Denote $E^c = \{e_i \mid e_i$ changes its place with e_1 through the cycle$\}$. Let $q = \max\{i \mid e_i \in E^c\}$.

Consider the two situations when e_q enters the base and when e_q leaves the base. The oriented cocircuit Y_1 as row e_1 of the first tableau and the oriented circuit X_1 as column e_1 of the second tableau have the following properties:

1. $e_1 \in Y_1^+$	1'. $e_1 \in X_1^-$
2. $e_q \in Y_1^-$	2'. $e_q \in X_1^+$
3. $e_j \in E - Y_1^-$ if $j < q$	3'. $e_j \in E - X_1^+$ if $j < q$.

Since the oriented circuits X_1 and Y_1 have common elements (e_1, e_q) so to fulfil orthogonality condition both their positive and negative part must have common elements. We know that the elements e_1, e_q and e_j if $j<q$ are in opposite parts of these circuits. So in order to prove our theorem one has to show only, that their homogeneous parts have no common element, that is the set $(X_1^+ \cap Y_1^+) \cup (X_1^- \cap Y_1^-)$ is empty. As we have seen above only the elements e_j if $j>q$ might be in these sets, but these elements are only in one of the circuits X_1 and Y_1 since these ones are not elements of E^c, that is an element e_j for $j>q$ was a basic or nonbasic element through the entire cycle. □

3. Connection with the original algorithm

For an arbitrary oriented matroid, given base B and tableau $T(B)$ the row of the pivot element remains unchanged or multiplied by (-1) – depending on the sign of the pivot element – through a pivot operation. This is obvious, since the same elements generate the oriented cocircuit of the pivot row. It may be necessary to multiply this row by (-1) in order to have $(+1)$ in the column of the new basic element.

Bland's [1] algorithm, which was generalized by a criss-cross type method in [6] is the following. Let us choose a base B such that $e_1 \notin B$. Choose the lowest indexed positive element (e_r) in the e_1 column of the tableau, then choose the lowest indexed negative element (e_k) of the e_r row. So we have the pivot element (τ_{rk}).

By the above remaks it is obvious that our new algorithm in two steps produces the same base as Bland's one. We change first e_1 by e_r and then e_r by e_1, since after the first pivoting, row e_1 of the tableau is identical to row e_r of the initial tableau as we have seen above.

The same connection with the dual of Bland's algorithm holds, if we start with a base B and $e_1 \in B$.

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