ON AN INFINITE-PERSON DYNAMIC COALITION GAME

ALI SHAMANDY* and ZOLTÁN VARGA**

* Dept. of Math., Mansoura University, Mansoura, Egypt
** Dept. of Applied Analysis, Eötvös Loránd University, 1088 Budapest, Múzeum krt. 6–8.

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In this paper cooperative solutions of infinite-person games with time-depending fuzzy coalitions are studied. Infinite-person games were introduced by Aumann in [1]. For their applications in economics we refer to Hildebrandt [2]. The idea of using fuzzy coalitions (i. e. coalitions with "rates of participation") is due to Aubin (see [3], [4] and [5]). Dynamic coalition models were considered in [6]. Cooperative solutions of N-person dynamic games were studied e. g. in [7].

In the present paper the community of players is modelled by a compact metric space. A finite subset of players is supposed to control the environment. The dynamics of the latter is described by a linear differential equation, and its final state determines the pay-off of each player. Under continuity assumptions on the pay-off functions, the existence of a cooperative solution for the whole infinite set of players is proved.

1. Let Ω be a compact metric space. The elements of Ω are interpreted as *players*. We suppose that the environment of the players is described by the differential equations

$$\dot{x} = Ax$$

with an appropriate matrix $A \in \mathbf{R}^{n\times n}$. Then we consider a finite subset

$$\Omega_N = \{\omega_1, \ldots, \omega_N\} \subset \Omega$$

of distinguished players which have the option to influence the environment. The influence of player ω_i is given by the equation

$$\dot{x} = Ax + b_i$$

where $b_i \in \mathbf{R}^n$ is fixed ($i \in \overline{1, N}$).

For a given $c \subset \mathcal{N} := \{1, \ldots, N\}$ the set of players

(1) $$\Omega_c := \{\omega_i \colon i \in c\}$$

is interpreted as a *coalition*. The influence of a coalition is supposed to be the superposition of the single influences:

$$\dot{x} = Ax + \sum_{i \in c} b_i. \tag{2}$$

For a more flexible model, we suppose that, instead of forming a "strict" coalition, each player in Ω_N may have a "rate of participation" $v_i \in [0,1]$ $(i \in \overline{1, N})$.

Definition 1. Any element $v \in U := [0, 1]^{\mathcal{N}}$ is called a *fuzzy coalition*. Any $v \in U_0 := \{0, 1\}^{\mathcal{N}}$ is called a *pure coalition*.

Clearly, there is a natural one-to-one correspondence between the pure coalitions and the sets (1). Moreover, let's consider both U and U_0 as subsets of $\mathbf{R}^N$. Then the set of fuzzy coalitions is nothing else than the convex hull of the set of pure coalitions.

Now fix $T \in \mathbf{R}_+$ and put

$$\mathcal{U} := \{u \in L_2^N[O, T],\ u(t) \in U \text{ for a.e. } t \in [O, T]\}.$$

The elements of U are interpreted as *time-depending fuzzy coalitions*.

We fix an initial state $x_0 \in \mathbf{R}^n$ and consider the mapping L: $L_2^N[O, T] \to \mathbf{R}^n$ which associates with each $u \in L_2^N[O, T]$ the point $x(T)$ of the solution of the following initial value problem

$$\dot{x} = Ax + \sum_{i=1}^{N} b_i u_i, \tag{3}$$

$$x(0) = x_0.$$

For a $u \in U$, (3) is interpreted as the dynamics of the enviroment under the influence of the time-depending fuzzy coalition u. Clearly, (3) is an extension of (2) in some sense.

2. Having set up the dynamics of the game, we turn to the formalization of the pay-off. Let

$$g\colon \Omega \times \mathbf{R}^n \to \mathbf{R}$$

be an upper semicontinuous function such that for every $y \in \mathbf{R}^n$ the function $g(\cdot, y)$ is lower semicontinuous. The function g is interpreted as follows. For every $(\omega, y) \in \Omega x \mathbf{R}^n$, $g(\omega, y)$ is the pay-off received by the player ω, provided that the process (3) has ended at y.

Let's define

$$G\colon \mathbf{R}^n \to C(\Omega),\ G(y) := g(\cdot, y);$$

$$F := \mathrm{G} \circ L|_U.$$

According to the above interpretations, the function F: $U \to C(\Omega)$ associates the resulting pay-off as a continuous function of the player with every time-depending fuzzy coalition $u \in U$.

Definition 2. The pair $\Gamma := (U, F)$ is called an *infinite-person dynamic coalition game.*

Definition 3. A $u_* \in \mathcal{U}$ is called a *cooperative solution* of the game Γ if for every $u \in U$ the inequalities

$$F(u)(\omega) \geq F(u_*)(\omega) \quad (\omega \in \Omega)$$

imply that

$$F(u)(\omega) = F(u_*)(\omega) \quad (\omega \in \Omega).$$

Remark. In terms of vector optimization, a cooperative solution u_* provides a maximal value of the function F with respect to the partial ordering of $C(\Omega)$ according to the closed convex cone K of nonnegative functions in $C(\Omega)$. (See e. g. [8].)

In the next section we shall prove the existence of a cooperative solution to the game Γ. To this end we shall need the following.

Lemma. *For any convex compact set* $V \in \mathbf{R}^n$ *the set*

$$\mathcal{V} := \{u \in L_2^N[O, T] : u(t) \in V \text{ for a. e. } t \in [O, T]\}$$

is weakly compact in $L_2^N\,[O, T]$.

Proof. It is obvious that $\mathcal{V}$ is convex and bounded in norm. First we prove that $\mathcal{V}$ is closed in the norm topology.

Assume that $u_n \in \mathcal{V}$ $(n \in \mathbf{N})$ and $\lim\,(u_n) = u_0 \in \mathcal{V}$ in the norm topology. We prove that $u_0 \in \mathcal{V}$. $V \subset \mathbf{R}^n$ being convex, closed and $V \neq \mathbf{R}^N$, V can be represented as the intersection of a countable set of closed support half-spaces. Thus, it is sufficient to show that for any $a \in \mathbf{R}^N$ and $\alpha \in \mathbf{R}$

$$V \subset H := \{z \in \mathbf{R}^N : \langle a, z \rangle \leq \alpha\}$$

implies that

$$u_0(t) \in H \text{ for a.e. } t \in [O, T]. \tag{4}$$

For the characteristic function $\chi_M \colon [O, T] \to \mathbf{R}$ of the set

$$M := \{t \in [0, T] : u_0(t) \notin H\}$$

we have

$$\int_0^T \chi_M \langle a, u_n \rangle \, d\lambda \leq \alpha \lambda(M) \quad (n \in \mathbf{N}),$$

where λ is the Lebesque measure in $[0, T]$. Since the sequence (u_n) also weakly converges u_0, we get

$$\lim \left(\int_M \langle a, u_n \rangle d\lambda \right) = \lim \left(\int_0^T \chi_M \langle a, u_n \rangle d\lambda \right) = \int_0^T \chi_M \langle a, u_0 \rangle d\lambda =$$

$$= \int_M \langle a, u_0 \rangle d\lambda \leq \alpha \, \lambda(M).$$

To prove (4) suppose the contrary: $\lambda(M)>0$. Then from

$$\langle a, u_0(t)\rangle > \alpha \quad (t \in M)$$

we obtain that

$$\int_M \langle a, u_0\rangle d\lambda > \alpha\lambda(M),$$

which contradicts to (5).

Consequently, $\mathcal{V}$ is closed in the norm topology. By its convexity $\mathcal{V}$ is also weakly closed. On the other hand, any weakly closed convex and norm-bounded set in a reflexive Banach space is weakly compact (see e. g. [9]). □

3. In this section we prove the following existence theorem.

Theorem. *The infinite-person dynamic coalition game Γ has a cooperative solution.*

Proof. According to Krein's theorem (see [10]), for any separable Banach space Z ordered by a closed convex cone $K \subset Z$, there exists a functional $p \in Z^*$ which is strictly positive in the sense that

$$\langle p, z\rangle > 0 \ (z \in K \setminus \{0\}).$$

Since Ω is a compact metric space, $C(\Omega)$ is separable. Applying Krein's theorem with $Z := C(\Omega)$ and

$$K := \{\varphi \in C(\Omega) \colon \varphi(\omega) \geq 0 \ \ (\omega \in \Omega)\},$$

also using Riesz's theorem, we obtain that there exists a Borel measure μ in Ω with the following properties: For any $\varphi \in C(\Omega)$

$$\varphi(\omega) \geq 0 (\omega \in \Omega) \Rightarrow \int_\Omega \varphi d\mu \geq 0,$$

$$\left.\begin{matrix} \forall\, \omega \in \Omega \colon \varphi(\omega) \geq 0 \\ \exists \bar{\omega} \in \Omega \colon \varphi(\bar{\omega}) > 0 \end{matrix}\right\} \Rightarrow \int_\Omega \varphi \, d\, \mu > 0. \tag{6}$$

Now we define

$$f \colon \mathcal{U} \to \mathbf{R}, \ f(u) := \int_\Omega F(u) d\mu.$$

Now we shall prove that the continuity assumptions on g in Section 2 imply the upper semicontinuity of f in the weak topology of $\mathcal{U}$. Hence, by the weak compactness of $\mathcal{U}$, the classical Weierstrass theorem guarantees the existence of a minimum point u_* of f. We shall see that u_* is the required solution to Γ.

Let's fix a $u_0 \in \mathcal{U}$. In order to prove the upper semicontinuity of f at u_0, we pick an arbitrary positive real number ε. We define

$$Q := L(\mathcal{U}), \ \ y_0 := L(u_0).$$

First we show that there exists a neighbourhood Q_{y_0} of y_0 in Q such that

$$(7) \qquad g(\omega, y) < g(\omega, y_0) + \varepsilon/\mu(\Omega) \ (y \in Q_{y_0}, \ \omega \in \Omega).$$

We fix an arbitrary $\omega_0 \in \Omega$. Then, by the upper semicontinuity of g and the lower semicontinuity of $g((\cdot, y_0)$ there exist an open neighbourhood Q_{ω_0} of y_0 in Q and an open neighbourhood Ω_{ω_0} of ω_0 such that for every $\omega \in \Omega_{\omega_0}$, $y \in Q_{\omega_0}$ we have

$$g(\omega, y) < g(\omega_0, y_0) + \varepsilon/2\mu(\Omega)$$
$$g(\omega, y_0) > g(\omega_0, y_0) - \varepsilon/2\mu(\Omega).$$

The family

$$P_{\omega_0} := \Omega_{\omega_0} \times Q_{\omega_0} \subset \Omega \times Q \quad (\omega_0 \in \Omega)$$

is an open covering of the compact subset $\Omega \times \{y_0\} \subset \Omega \times Q$. Hence there exist a $k \in \mathbf{N}$ and elements

$$\omega^1, \ \ldots, \ \omega^k \in \Omega$$

such that

$$\Omega \times \{y_0\} \subset \bigcup_{j=1}^{k} P_{\omega j}.$$

The set $Q_{y_0} := \bigcap_{j=1}^{k} Q_{\omega j}$ is a neighbourhood of y_0 in Q such that for every $\omega \in \Omega$ and $y \in Q_{y_0}$ there exists a $j \in \overline{1, k}$ with $(\omega, y) \in P_{\omega_j}$ and

$$g(\omega, y) - g(\omega, y_0) = g(\omega, y) - g(\omega^j, y_0) +$$
$$+ g(\omega^j, y_0) - g(\omega, y_0) < \varepsilon/2\mu(\Omega) + \varepsilon/2\mu(\Omega) = \varepsilon/\mu(\Omega).$$

Now, from the variation of parameters formula, it follows easily that the operator L is continuous in the weak topology of $L_2^N[0, T]$. Therefore there exists a neighbourhood $\mathcal{U}_{u_0}$ of u_0 in $\mathcal{U}$ such that

$$L(u) \in Q_{y_0} \quad (u \in \mathcal{U}_{u_0}).$$

Hence, by (7), we get that for any $u \in \mathcal{U}_{u_0}$ and $\omega \in \Omega$

$$F(u)(\omega) = G(L(u))(\omega) = g(\omega, L(u)) <$$
$$< g(\omega, L(u_0)) + \varepsilon/\mu(\Omega) = F(u_0)(\omega) + \varepsilon/\mu(\Omega).$$

By integration with respect to the measure μ, we obtain that for every $u \in \mathcal{U}_{u_0}$

$$f(u) = \int_\Omega F(u)\, d\mu < \int_\Omega F(u_0)\, d\mu + \varepsilon = f(u_0) + \varepsilon$$

that is f is upper semicontinuous in the weak topology of $L_2^N[O, T]$.

Since $\mathcal{U}$ is compact in the latter topology, by the Weierstrass theorem it follows that there exists a $u_* \in \mathcal{U}$ such that

$$f(u) \leq f(u_*) \quad (u \in \mathcal{U}). \tag{8}$$

Finally, we show that u_* is a cooperative solution of Γ. Indeed, let's suppose the contrary. Then, by Definition 3, there exists a $u \in \mathcal{U}$ such that

$$F(u)(\omega) \geq F(u_*)(\omega) \quad (\omega \in \Omega)$$

and for some $\bar{\omega} \in \Omega$

$$F(u)(\bar{\omega}) > F(u_*)(\bar{\omega}).$$

Since μ has the property (6), taking $\varphi := F(u) - F(u_*)$ we get

$$f(u) - f(u_*) = \int_\Omega [F(u) - F(u_*)] d\mu > 0$$

in contradiction with (8). □

Remark. It is easy to see that the above theorem also holds for games with time-depending dynamics, i. e. in case in (3) A and b_i ($i \in \overline{1, N}$) are, say, continuous functions.

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